Wavelets and Multiresolution Processing

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Multiresolution Analysis (MRA)

- A scaling function is used to create a series of approximations of a function or image, each differing by a factor of 2 from its neighboring approximations.
- Additional functions called wavelets are then used to encode the difference in information between adjacent approximations.

Series Expansions

- Express a signal $f(x)$ as

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

expansion functions

expansion coefficients

- If the expansion is unique, the $\varphi_k(x)$ are called basis functions, and the expansion set $\{\varphi_k(x)\}$ is called a basis

Series Expansions

- All the functions expressible with this basis form a function space which is referred to as the closed span of the expansion set

$$V = \overline{\underset{k}{Span}\{\varphi_k(x)\}}$$

- If $f(x) \in V$, then $f(x)$ is in the closed span of $\{\varphi_k(x)\}$ and can be expressed as

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

Orthonormal Basis

- The expansion functions form an orthonormal basis for *V*

$$\left\langle \varphi_j(x), \varphi_k(x) \right\rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

- The basis and its dual are equivalent, i.e.,

  $\varphi_k(x) = \tilde{\varphi}_k(x)$ and

$$\alpha_k = \left\langle \varphi_k(x), f(x) \right\rangle = \int \varphi_k^*(x) f(x)\, dx$$

Scaling Functions

- Consider the set of expansion functions composed of integer translations and binary scalings of the real square-integrable function $\varphi(x)$ defined by

$$\{\varphi_{j,k}(x)\} = \{2^{j/2}\varphi(2^j x - k)\}$$

for all $j,k \in \mathbb{Z}$ and $\varphi(x) \in L^2(\mathbb{R})$

- By choosing the scaling function $\varphi(x)$ wisely, $\{\varphi_{j,k}(x)\}$ can be made to span $L^2(\mathbb{R})$

$$\left\{\varphi_{j,k}(x)\right\} = \left\{2^{j/2}\varphi(2^j x - k)\right\}$$

- Index *k* determines the position of $\varphi_{j,k}(x)$ along the *x*-axis, index *j* determines its width; $2^{j/2}$ controls its height or amplitude.
- By restricting *j* to a specific value $j = j_o$ the resulting expansion set $\left\{\varphi_{j_o,k}(x)\right\}$ is a subset of $\left\{\varphi_{j,k}(x)\right\}$
- One can write $V_{j_o} = \overline{\underset{k}{Span}\left\{\varphi_{j_o,k}(x)\right\}}$

Example: The Haar Scaling Function

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & otherwise \end{cases}$$

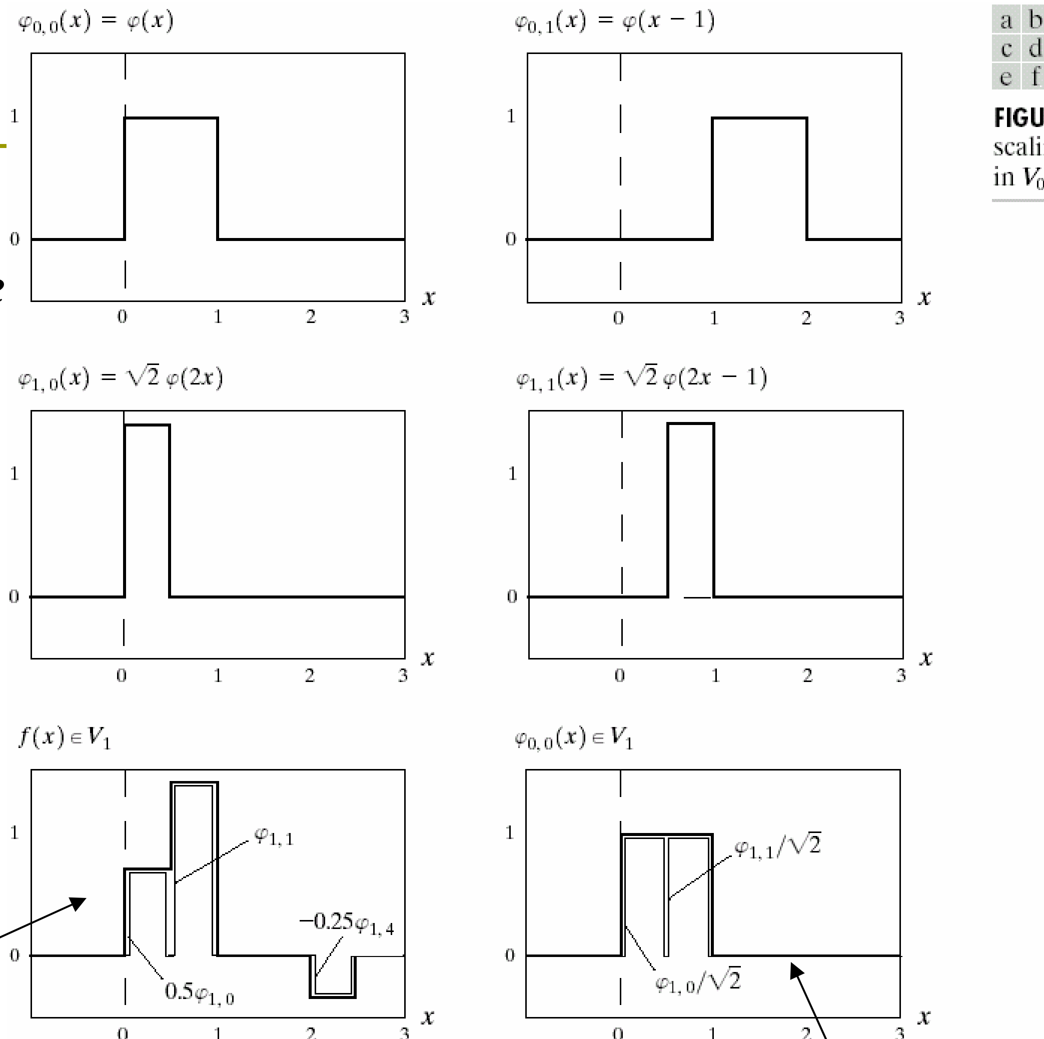


FIGURE 7.9 Haar scaling functions in V_0 in V_1.

$$V_0 \subset V_1$$

$$f(x) = 0.5\varphi_{1,0}(x) + \varphi_{1,1}(x) - 0.25\varphi_{1,4}(x)$$

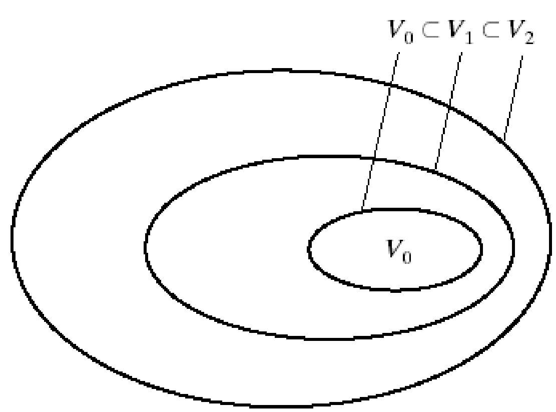
$$\varphi_{0,k}(x) = \frac{1}{\sqrt{2}}\varphi_{1,2k}(x) + \frac{1}{\sqrt{2}}\varphi_{1,2k+1}(x)$$

MRA Requirements

1. *The scaling function is orthogonal to its integer translates*
2. *The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales:*

$$V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{\infty}$$

FIGURE 7.10 The nested function spaces spanned by a scaling function.



Wavelet Functions

- Given a scaling function which satisfies the MRA requirements, one can define a **wavelet function** $\psi(x)$ which, together with its integer translates and binary scalings, *spans the difference between any two adjacent scaling subspaces* V_j *and* V_{j+1}

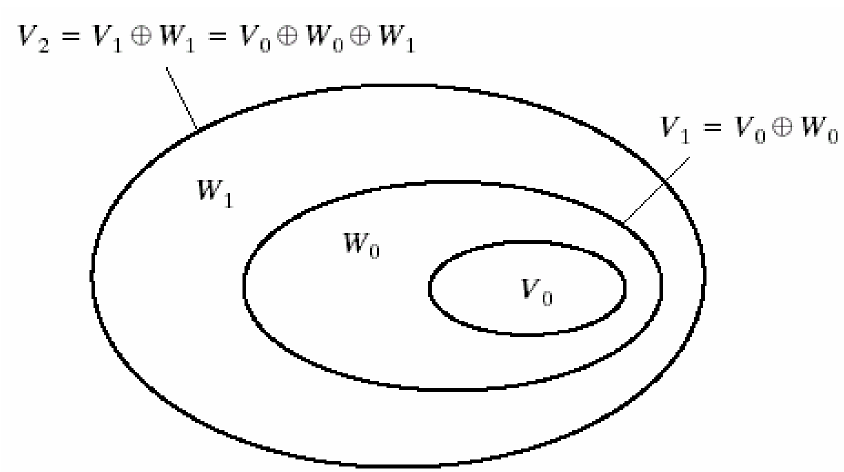


FIGURE 7.11 The relationship between scaling and wavelet function spaces.

Wavelet Functions

- Define the wavelet set

$$\left\{\psi_{j,k}(x)\right\} = \left\{2^{j/2}\psi(2^j x - k)\right\}$$

for all $k \in \mathbb{Z}$ that spans the W_j spaces

- We write

and, if

$$W_j = \overline{\underset{k}{Span}\left\{\psi_{j,k}(x)\right\}}$$

$$f(x) \in W_j$$

$$f(x) = \sum_k \alpha_k \psi_{j,k}(x)$$

Orthogonality: $V_{j+1} = V_j \oplus W_j$

- This implies that

$$\langle \varphi_{j,k}(x), \psi_{j,l}(x) \rangle = 0$$

for all appropriate $j,k,l \in \mathbb{Z}$

- We can write

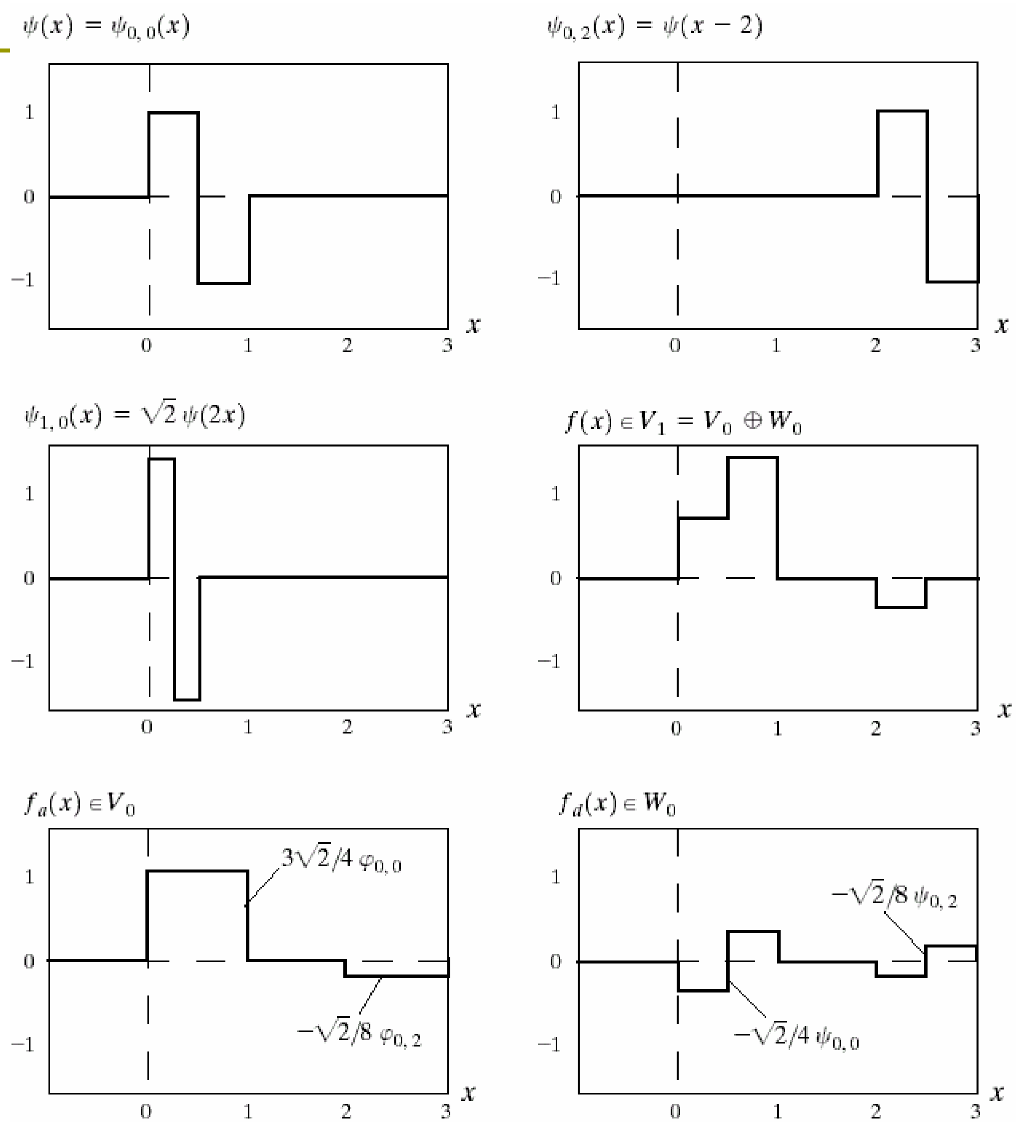
$$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots$$

and also

$$L^2(\mathbb{R}) = \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

(no need for scaling functions, only wavelets!)

Example: Haar Wavelet Functions in W_0 and W_1



a b
c d
e f

FIGURE 7.12 Haar wavelet functions in W_0 and W_1.

$$f(x) = f_a(x) + f_d(x)$$

low frequencies ($f_a(x)$)

high frequencies ($f_d(x)$)

Wavelet Series Expansions

- A function $f(x) \in L^2(\mathbb{R})$ can be expressed as

$$f(x) = \sum_k c_{j_0}(k)\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_k d_j(k)\psi_{j,k}(x)$$

approximation or scaling coefficients

detail or wavelet coefficients

$$c_{j_0}(k) = \langle f(x), \varphi_{j_0,k}(x) \rangle$$

$$d_j(k) = \langle f(x), \psi_{j,k}(x) \rangle$$

$$L^2(\mathbb{R}) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \cdots$$

Example: The Haar Wavelet Series Expansion of $y=x^2$

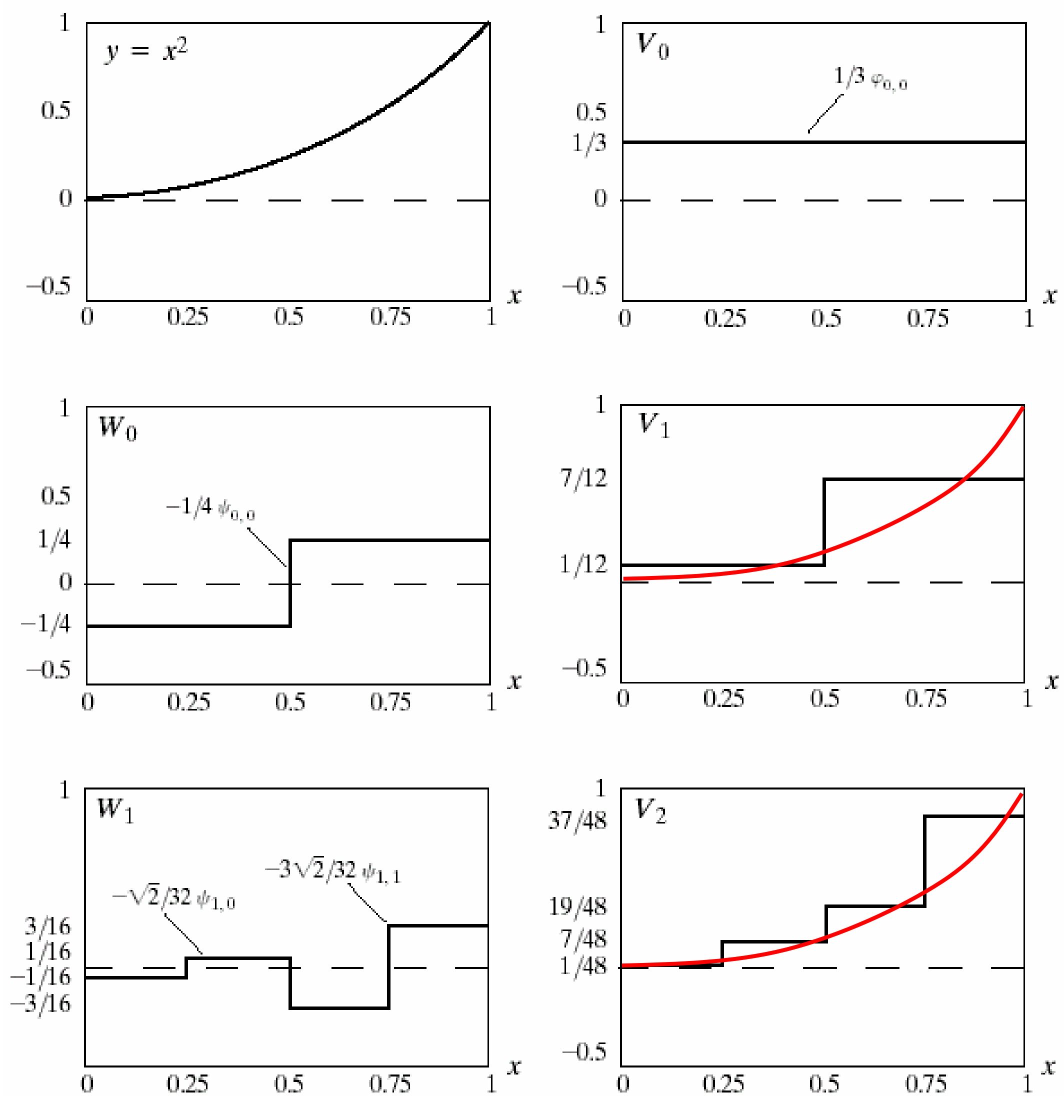
- Consider $y = \begin{cases} x^2 & 0 \le x < 1 \\ 0 & otherwise \end{cases}$
- If $j_0 = 0$, the expansion coefficients are

$$c_0(0) = \int_0^1 x^2 \varphi_{0,0}(x)\,dx = \frac{1}{3} \qquad d_0(0) = \int_0^1 x^2 \psi_{0,0}(x)\,dx = -\frac{1}{4}$$

$$d_1(0) = \int_0^1 x^2 \psi_{1,0}(x)\,dx = -\frac{\sqrt{2}}{32} \qquad d_1(1) = \int_0^1 x^2 \psi_{1,1}(x)\,dx = -\frac{3\sqrt{2}}{32}$$

$$y = \underbrace{\underbrace{\underbrace{\frac{1}{3}\varphi_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4}\psi_{0,0}(x)\right]}_{W_0}}_{V_1 = V_0 \oplus W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32}\psi_{1,0}(x) - \frac{3\sqrt{2}}{32}\psi_{1,1}(x)\right]}_{W_1}}_{V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1} + \cdots$$

Example: The Haar Wavelet Series Expansion of $y=x^2$



The Discrete Wavelet Transform (DWT)

- Let $f(x),\ x = 0,1,\ldots,M-1$ denote a discrete function
- Its DWT is defined as

approximation coefficients

$$W_\varphi(j_0,k) = \frac{1}{\sqrt{M}}\sum_x f(x)\varphi_{j_0,k}(x)$$

detail coefficients

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_x f(x)\psi_{j,k}(x) \quad j \ge j_0$$

$$f(x) = \frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\varphi_{j_0,k}(x) + \frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\psi_{j,k}(x)$$

- Let $j_o = 0$ and $M = 2^J$ so that $\begin{cases} x = 0,1,\ldots,M-1, \\ j = 0,1,\ldots,J-1, \\ k = 0,1,\ldots,2^j-1 \end{cases}$

Example: Computing the DWT

- Consider the discrete function

$$f(0)=1, f(1)=4, f(2)=-3, f(3)=0$$

- It is $M=4=2^2 \longrightarrow J=2$
- The summations are performed over

  $x=0,1,2,3$ and $k=0$ for $j=0$ and

  $k=0,1$ for $j=1$
- Use the Haar scaling and wavelet functions

Example: Computing the DWT

$$W_{\varphi}(0,0)=\frac{1}{2}\sum_{x=0}^{3} f(x)\varphi_{0,0}(x)=\frac{1}{2}\left[1\cdot 1+4\cdot 1-3\cdot 1+0\cdot 1\right]=1$$

$$W_{\psi}(0,0)=\frac{1}{2}\sum_{x=0}^{3} f(x)\psi_{0,0}(x)=\frac{1}{2}\left[1\cdot 1+4\cdot 1-3\cdot(-1)+0\cdot(-1)\right]=4$$

$$W_{\psi}(1,0)=\frac{1}{2}\sum_{x=0}^{3} f(x)\psi_{1,0}(x)=\frac{1}{2}\left[1\cdot\sqrt{2}+4\cdot(-\sqrt{2})-3\cdot 0+0\cdot 0\right]=-1.5\sqrt{2}$$

$$W_{\psi}(1,1)=\frac{1}{2}\sum_{x=0}^{3} f(x)\psi_{1,1}(x)=\frac{1}{2}\left[1\cdot 0+4\cdot 0-3\cdot\sqrt{2}+0\cdot(-\sqrt{2})\right]=-1.5\sqrt{2}$$

Example: Computing the DWT

- The DWT of the 4-sample function relative to the Haar wavelet and scaling functions thus is

$$\left\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\right\}$$

- The original function can be reconstructed as

$$f(x) = \frac{1}{2}\Big[W_\varphi(0,0)\varphi_{0,0}(x) + W_\psi(0,0)\psi_{0,0}(x) +$$

for $$W_\psi(1,0)\psi_{1,0}(x) + W_\psi(1,1)\psi_{1,1}(x)\Big]$$

$$x = 0, 1, 2, 3$$

Wavelet Transform in 2-D

- In 2-D, one needs one scaling function

$$\varphi(x,y)=\varphi(x)\varphi(y)$$

and three wavelets

$$\begin{cases} \psi^{H}(x,y)=\psi(x)\varphi(y) & \text{•detects horizontal details} \\ \psi^{V}(x,y)=\varphi(x)\psi(y) & \text{•detects vertical details} \\ \psi^{D}(x,y)=\psi(x)\psi(y) & \text{•detects diagonal details} \end{cases}$$

- $\varphi(.)$ is a 1-D scaling function and is its corresponding wavelet $\psi(.)$

2-D DWT: Definition

- Define the scaled and translated basis functions

$$\varphi_{j,m,n}(x,y) = 2^{j/2}\varphi(2^j x - m, 2^j y - n)$$

$$\psi^i_{j,m,n}(x,y) = 2^{j/2}\psi^i(2^j x - m, 2^j y - n), \quad i = \{H, V, D\}$$

- Then

$$W_\varphi(j_0,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\varphi_{j_0,m,n}(x,y)$$

$$W^i_\psi(j,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\psi^i_{j,m,n}(x,y), \quad i = \{H, V, D\}$$

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_m\sum_n W_\varphi(j_0,m,n)\varphi_{j_0,m,n}(x,y)$$

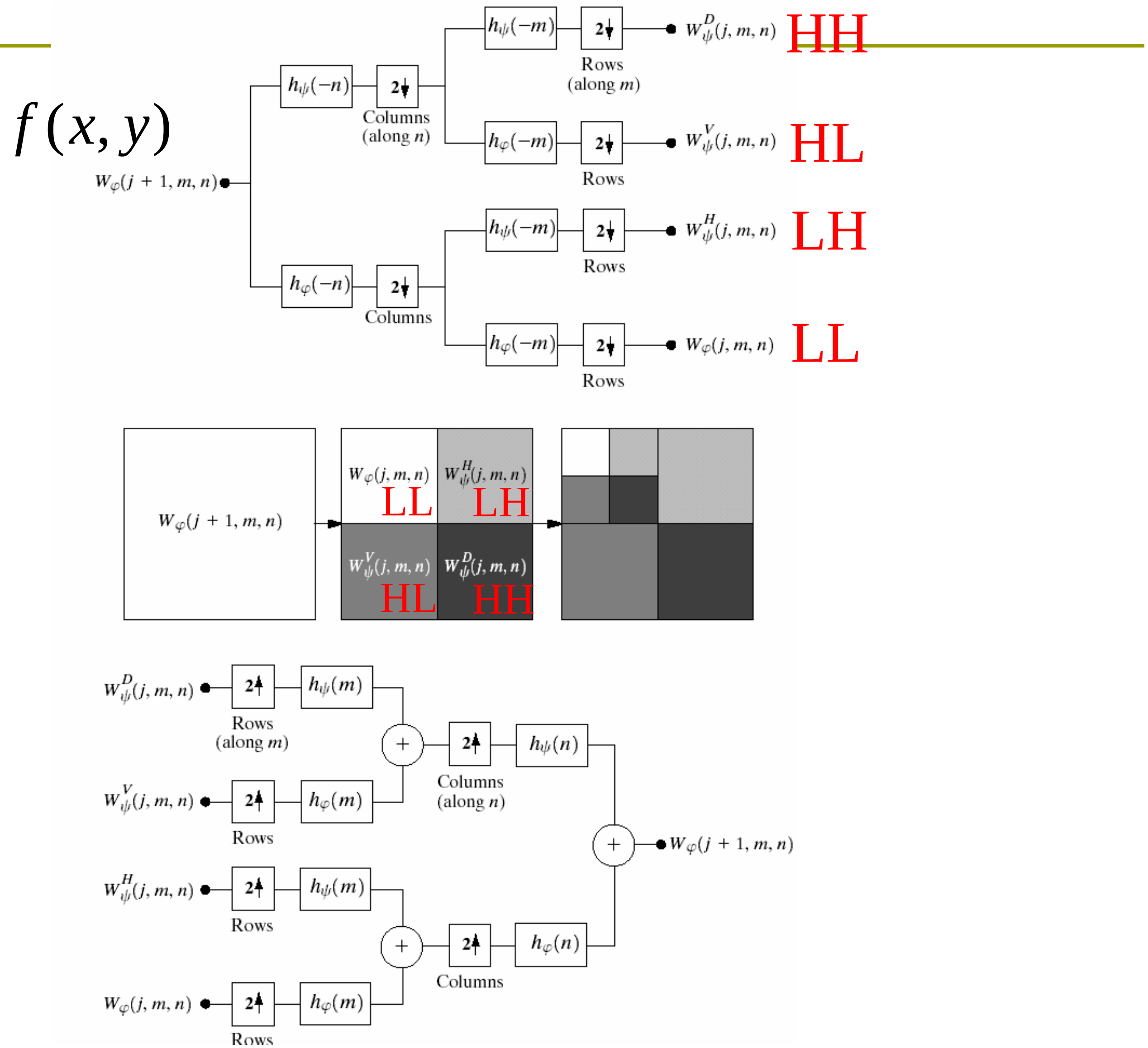
$$+\frac{1}{\sqrt{MN}}\sum_{i=H,V,D}\sum_{j=j_0}^{\infty}\sum_m\sum_n W^i_\psi(j,m,n)\psi^i_{j,m,n}(x,y)$$

Filter bank implementation of 2-D wavelet

analysis FB

resulting decomposition

synthesis FB



Example: A Three-Scale FWT

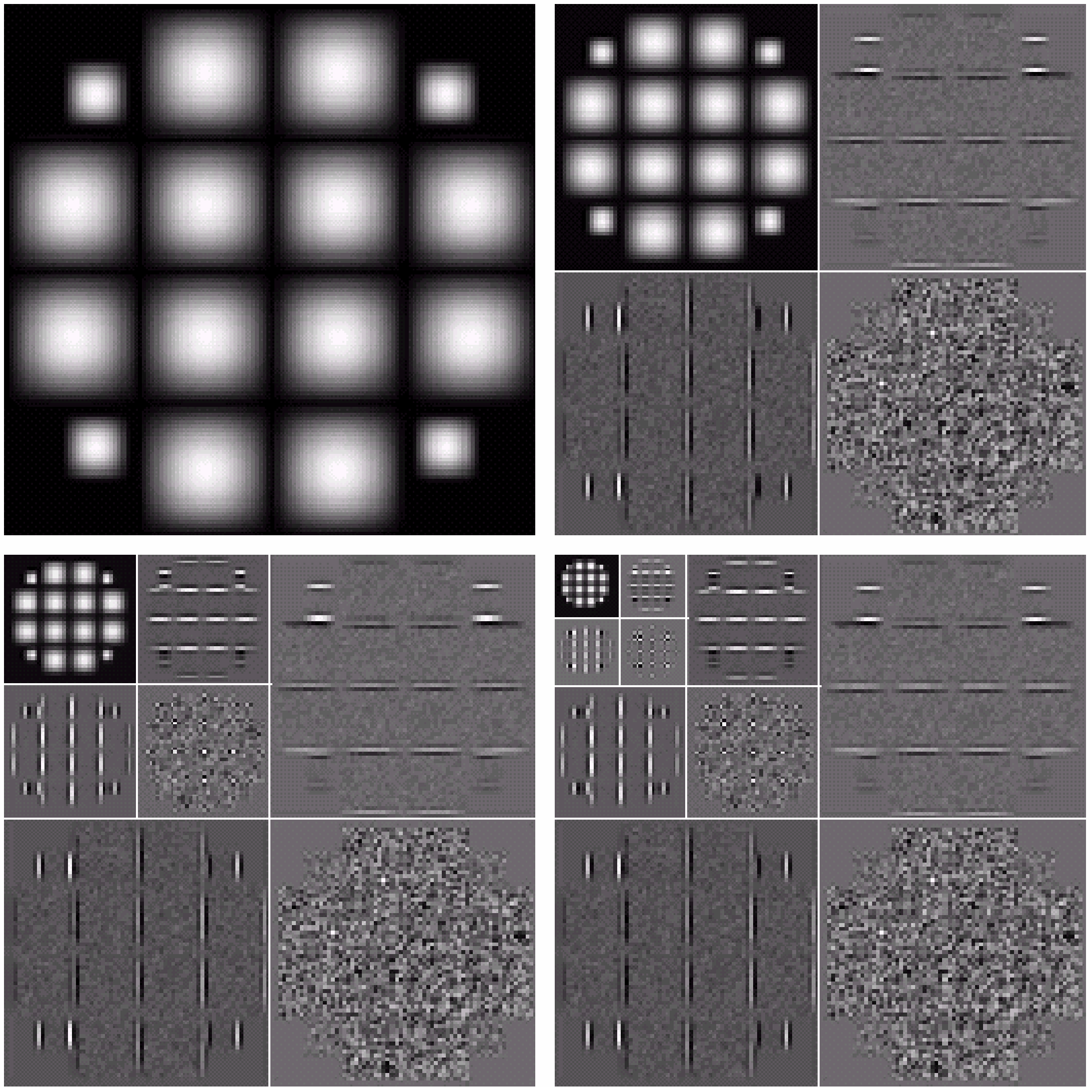
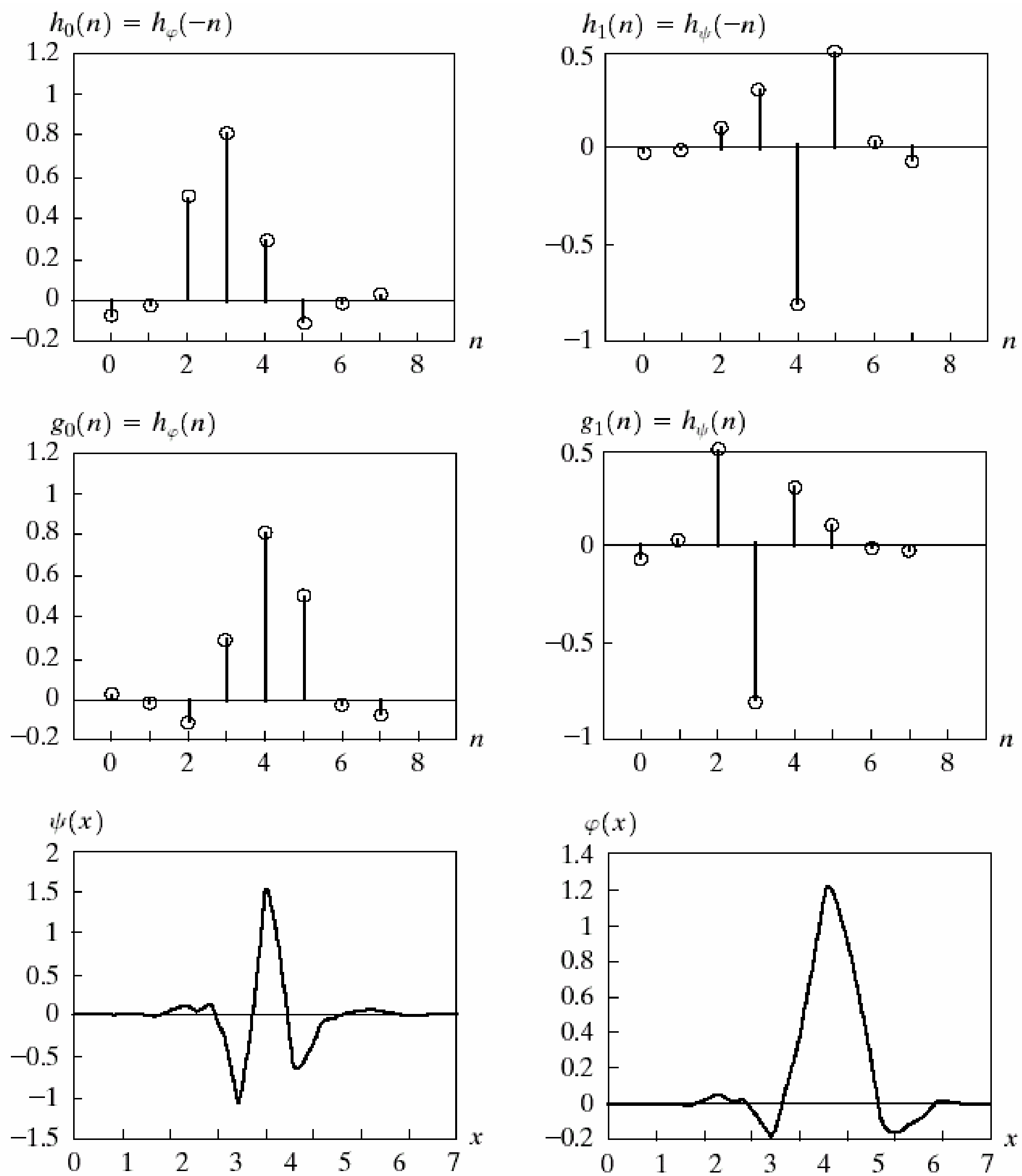


FIGURE 7.23 A three-scale FWT.

Analysis and Synthesis Filters

a b
c d
e f
g

FIGURE 7.24 Fourth-order symlets: (a)–(b) decomposition filters; (c)–(d) reconstruction filters; (e) the one-dimensional wavelet; (f) the one-dimensional scaling function; and (g) one of three two-dimensional wavelets, $\psi^H(x, y)$.



analysis filters

synthesis filters

wavelet function

scaling function

Want to Learn More About Wavelets?

- "*An Introduction to Wavelets*," by Amara Graps
- Amara's Wavelet Page (with many links to other resources) http://www.amara.com/current/wavelet.html
- "*Wavelets for Kids*," (A Tutorial Introduction), by B. Vidakovic and P. Mueller
- Gilbert Strang's tutorial papers from his MIT webpage

  http://www-math.mit.edu/~gs/
- ***Wavelets and Subband Coding*****,** by Jelena Kovacevic and Martin Vetterli, Prentice Hall, 2000.