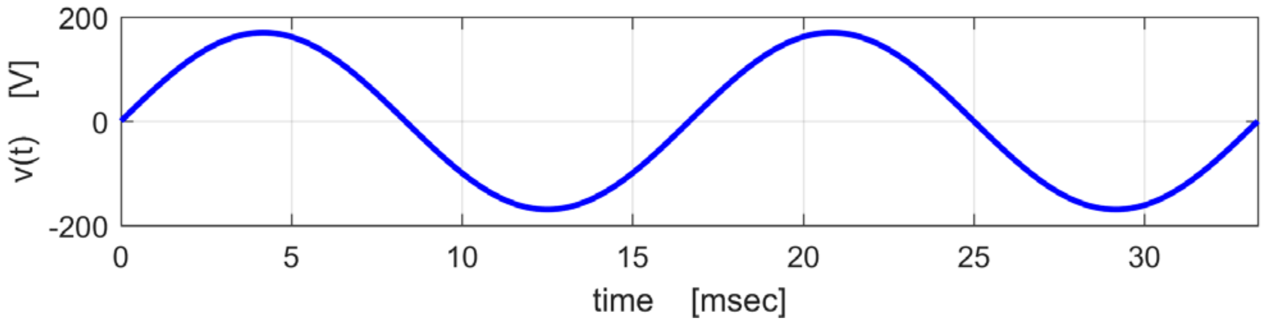
SECTION 1: SINUSOIDAL STEADY-STATE ANALYSIS

ENGR 202 – Electrical Fundamentals II

Sinusoids

Sinusoidal Signals

- ***Sinusoidal*** signals are of particular interest in the field of electrical engineering



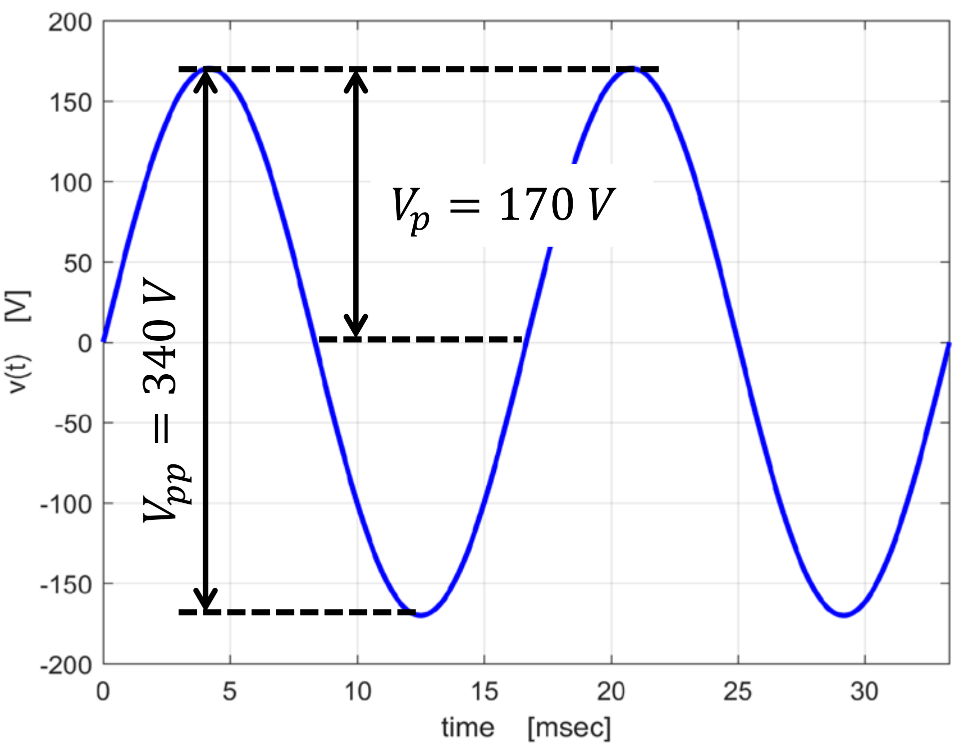
$$v(t) = V_p \cos(\omega t + \phi) = V_p \cos(2\pi \cdot f \cdot t + \phi)$$

- Sinusoidal signals defined by three parameters:
  - ***Amplitude***: V_p
  - ***Frequency***: ω or f
  - ***Phase***: ϕ

Amplitude

- ***Amplitude*** of a sinusoid is its ***peak*** voltage, V_p
- ***Peak-to-peak voltage***, V_{pp}, is twice the amplitude
  - $V_{pp} = 2V_p$
  - $V_{pp} = V_{max} - V_{min}$

$$v(t) = V_p \cdot \sin(\omega t + \phi) = V_p \cdot \sin(2\pi f t + \phi)$$



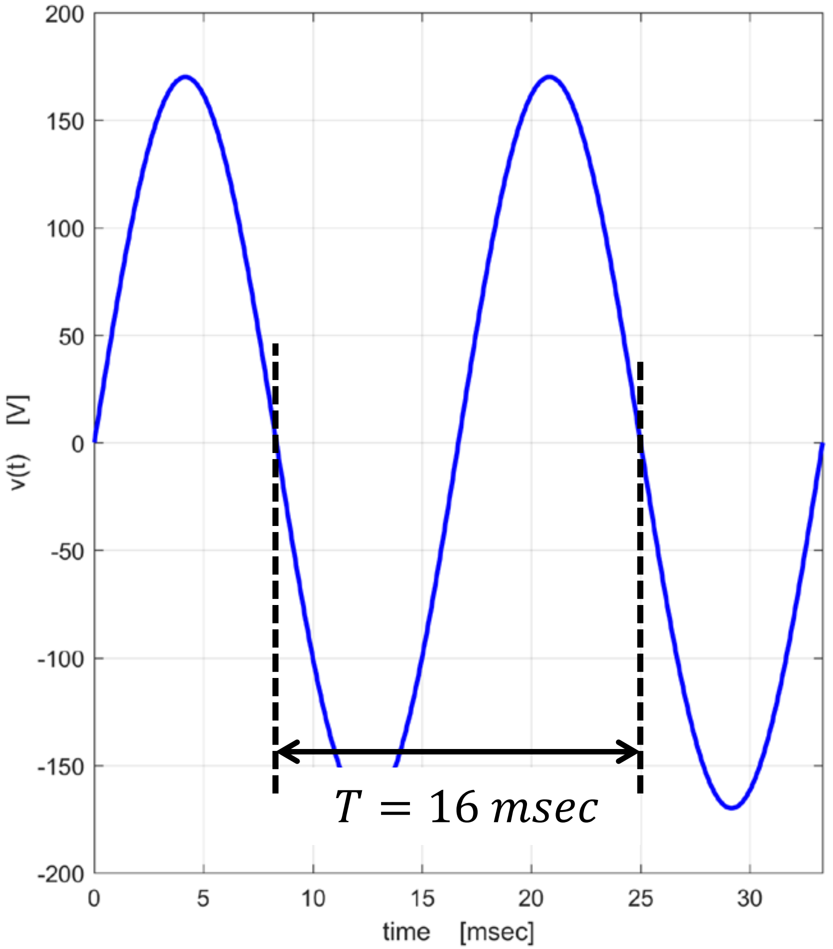
Frequency

- ***Period*** (T)
  - Duration of one cycle
- ***Frequency*** (f)
  - Number of periods per second

$$f = \frac{1}{T}$$

- ***Ordinary frequency***, f
  - Units: hertz (Hz), sec^{-1}, cycles/sec
- ***Angular frequency***, ω
  - Units: rad/sec

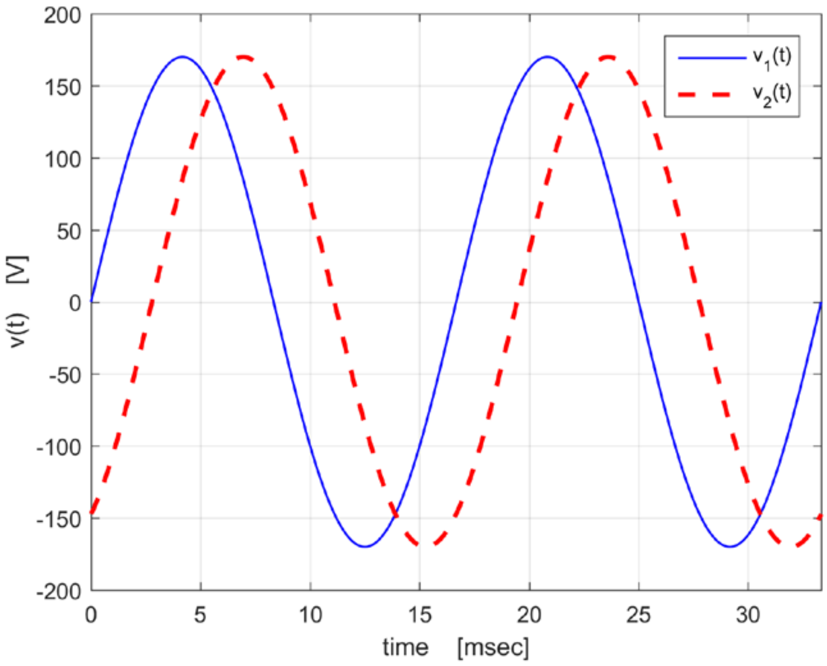
$$\omega = 2\pi f, \quad f = \frac{\omega}{2\pi}$$



Phase

- ***Phase***
  - Angular constant in signal expression, ϕ

$$v(t) = V_p \sin(\omega t + \phi)$$

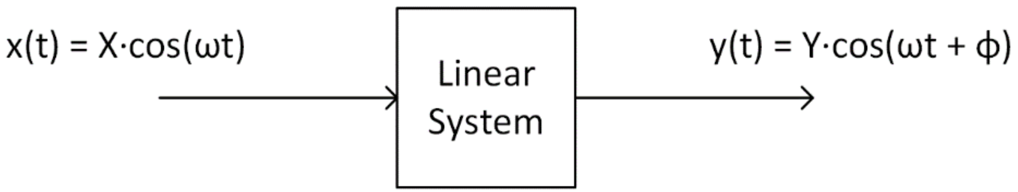


- Requires a time reference
  - Interested in relative, not absolute, phase
- Here,
  - $v_1(t)$ leads $v_2(t)$
  - $v_2(t)$ lags $v_1(t)$
- Units: radians
  - Not technically correct, but OK to express in degrees, e.g.:

$$v(t) = 170\,V \sin(2\pi \cdot 60Hz \cdot t + 34°)$$

Sinusoidal Steady-State Analysis

- Often interested in the response of linear systems to ***sinusoidal inputs***
  - Voltages and currents in electrical systems
  - Forces, torques, velocities, etc. in mechanical systems
- For ***linear systems*** excited by a sinusoidal input
  - Output is sinusoidal
    - ***Same frequency***
    - In general, ***different amplitude***
    - In general, ***different phase***



- We can simplify the analysis of linear systems by using ***phasor representation*** of sinusoids

Phasors

- ***Phasor***
  - A ***complex number*** representing the ***amplitude*** and ***phase*** of a sinusoidal signal
  - Frequency is not included
    - Remains constant and is accounted for separately
    - System characteristics (frequency-dependent) evaluated at the frequency of interest as first step in the analysis
- Phasors are ***complex numbers***
  - Before applying phasors to the analysis of electrical circuits, we'll first review the properties of complex numbers

Complex Numbers

Complex Numbers

- A complex number can be represented as

$$z = x + jy$$

  - x: real part (a real number)
  - y: imaginary part (a real number)
  - $j = \sqrt{-1}$ is the imaginary unit

- Complex numbers can be represented three ways:
  - ***Cartesian*** form: $z = x + jy$
  - ***Polar*** form: $z = r\angle\phi$
  - ***Exponential*** form: $z = re^{j\phi}$

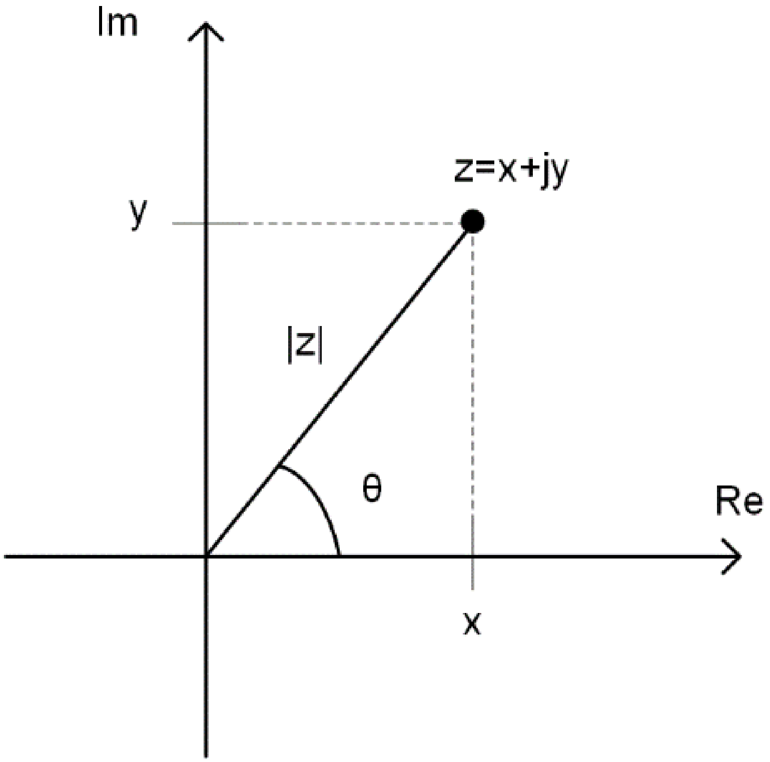
Complex Numbers as Vectors

- A complex number can be represented as a ***vector in the complex plane***
- Complex plane
  - ***Real axis*** – horizontal
  - ***Imaginary axis*** – vertical
- A vector from the origin to z
  - Real part, x
  - Imaginary part, y

$$z = x + jy$$

  - Vector has a ***magnitude***, r
  - And an ***angle***, θ

$$z = r\angle\theta$$



Cartesian Form ↔ Polar Form

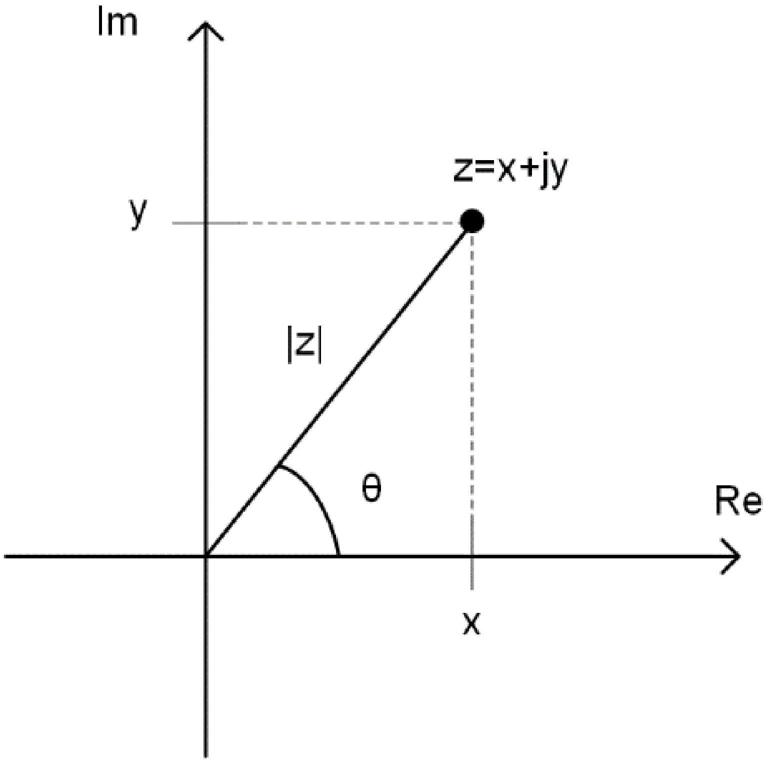
- Cartesian form → Polar form

$$z = x + jy = r\angle\theta$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \arg(z) = \angle z$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$



- Polar form → Cartesian form

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

Complex Numbers – Addition/Subtraction

- ***Addition and subtraction*** of complex numbers
  - Best done in ***Cartesian*** form
  - Real parts add/subtract
  - Imaginary parts add/subtract
- For example:

$$z_1 = x_1 + jy_1$$

$$z_2 = x_2 + jy_2$$

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Complex Numbers – Multiplication/Division

- ***Multiplication and division*** of complex numbers
  - Best done in ***polar*** form
  - Magnitudes multiply/divide
  - Angles add/subtract
- For example:

$$z_1 = r_1\angle\theta_1$$

$$z_2 = r_2\angle\theta_2$$

$$z_1 \cdot z_2 = r_1 r_2\angle(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

Complex Conjugate

- ***Conjugate*** of a complex number
  - Number that results from ***negating the imaginary part***

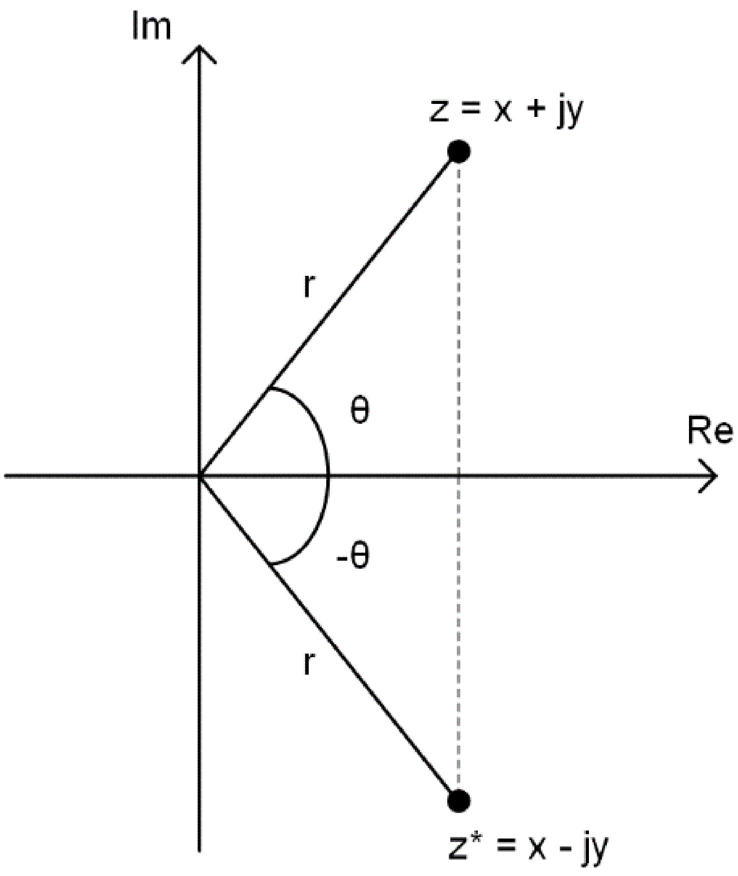
$$z = x + jy$$

$$z^* = x - jy$$

  - Or, equivalently, from ***negating the angle***

$$z = r\angle\theta$$

$$z^* = r\angle -\theta$$



Complex Fractions

- Multiplying a number by its complex conjugate yields the ***squared magnitude*** of that number
  - A ***real number***

$$z \cdot z^* = (x + jy)(x - jy) = x^2 + y^2$$

$$z \cdot z^* = r\angle\theta \cdot r\angle - \theta = r^2\angle\theta - \theta = r^2$$

- ***Rationalizing the denominator*** of a complex fraction:
  - Multiply numerator and denominator by the ***complex conjugate of the denominator***

$$z = \frac{x_1 + jy_1}{x_2 + jy_2} \cdot \frac{x_2 - jy_2}{x_2 - jy_2}$$

$$z = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\frac{(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

Complex Fractions

- ***Fractions*** or ***ratios*** are, of course, simply division
  - Very common form, so worth emphasizing
- ***Magnitude*** of a ratio of complex numbers

$$z = \frac{z_1}{z_2} \quad \rightarrow \quad |z| = \frac{|z_1|}{|z_2|}$$

- ***Angle*** of a ratio of complex numbers

$$z = \frac{z_1}{z_2} \quad \rightarrow \quad \angle z = \angle z_1 - \angle z_2$$

- ***Calculators and complex numbers***
  - Manipulation of complex numbers by hand is tedious and error-prone
  - Your calculators can perform complex arithmetic
  - They will operate in both Cartesian and polar form, and will convert between the two
  - Learn to use them – correctly

Phasors

Euler's Identity

- Fundamental to phasor notation is ***Euler's identity***:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

where j is the imaginary unit, and ω is angular frequency

- It follows that

$$\cos(\omega t) = Re\{e^{j\omega t}\}$$

$$\sin(\omega t) = Im\{e^{j\omega t}\}$$

and

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Phasors

- Consider a sinusoidal voltage

$$v(t) = V_p \cos(\omega t + \phi)$$

- Using Euler's identity, we can represent this as

$$v(t) = Re\left\{V_p e^{j(\omega t + \phi)}\right\} = Re\left\{V_p e^{j\phi} e^{j\omega t}\right\}$$

  where
  - V_p represents magnitude
  - $e^{j\phi}$ represents phase
  - $e^{j\omega t}$ represents a sinusoid of frequency ω

- Grouping the first two terms together, we have

$$v(t) = Re\left\{\mathbf{V} e^{j\omega t}\right\}$$

  where $\mathbf{V}$ is the phasor representation of $v(t)$

Phasors

$$v(t) = Re\{\mathbf{V}e^{j\omega t}\}$$

- The phasor representation of $v(t)$

$$\mathbf{V} = V_p e^{j\phi}$$

  - A representation of magnitude and phase only
  - Time-harmonic portion ($e^{j\omega t}$) has been dropped

Time-domain representation:	⟷	**Phasor-domain representation:**
$v(t) = V_p \sin(\omega t + \phi)$		$\mathbf{V} = V_p e^{j\phi} = V_p \angle \phi$

- Phasors greatly simplify sinusoidal steady-state analysis
  - Messy trigonometric functions are eliminated
  - Differentiation and integration transformed to algebraic operations

Voltage & Current in the Phasor Domain

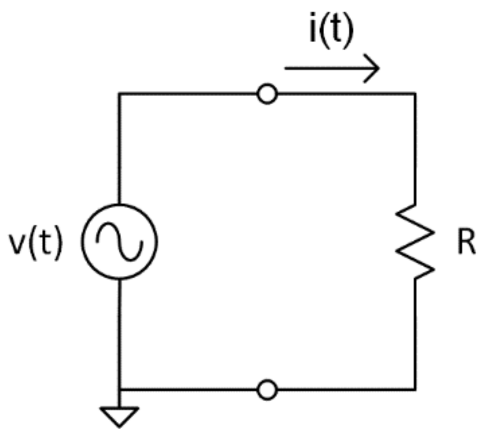
- We will use phasors to simplify analysis of electrical circuits
  - Need an understanding of electrical component behavior in the phasor domain
  - Relationships between ***voltage phasors*** and ***current phasors*** for Rs, Ls, and Cs

---

- ***Resistor***
  - Voltage across a resistor given by

$$v(t) = i(t)R$$

$$i(t) = I_p \cos(\omega t + \phi)$$



  - Converting to phasor form

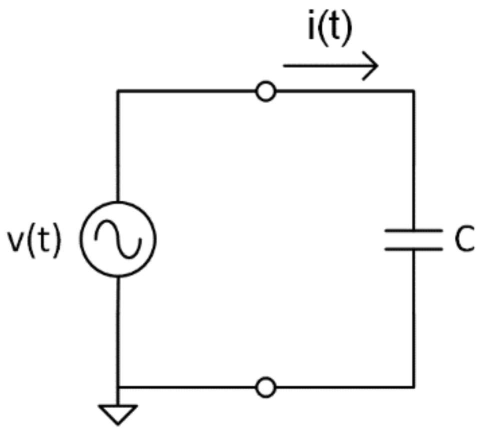
$$\mathbf{V} = \left(I_p e^{j\phi}\right)R$$

$$\boxed{\mathbf{V} = \mathbf{I}R} \qquad \boxed{\mathbf{I} = \frac{\mathbf{V}}{\mathrm{R}}}$$

  - ***Ohm's law*** in phasor form

V-I Relationships in the Phasor Domain

- ***Capacitor***
  - Current through the capacitor given by



$$i(t) = C\frac{dv}{dt}$$

$$i(t) = C\frac{d}{dt}\left[V_p\cos(\omega t + \phi)\right]$$

$$i(t) = -\omega C V_p \sin(\omega t + \phi)$$

  - Applying a trig identity:

$$-\sin(A) = \cos(A + 90°)$$

gives

$$i(t) = \omega C V_p \cos(\omega t + \phi + 90°)$$

  - Converting to ***phasor form***

$$\mathbf{I} = \omega C V_p e^{j(\phi + 90°)} = \omega C V_p e^{j\phi} e^{j90°}$$

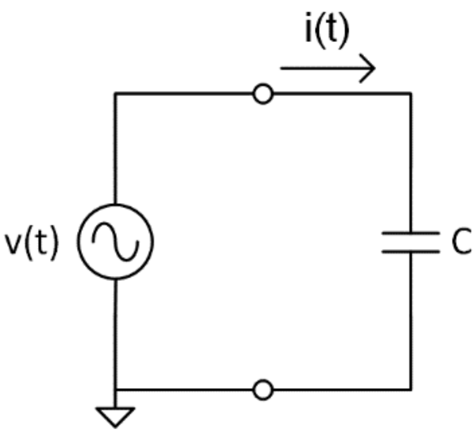
V-I Relationships - Capacitor

- Current phasor

$$\mathbf{I} = \omega C V_p e^{j(\phi+90°)} = \omega C V_p e^{j\phi} e^{j90°}$$

  - Voltage phasor is

$$\mathbf{V} = V_p e^{j\phi}$$



so

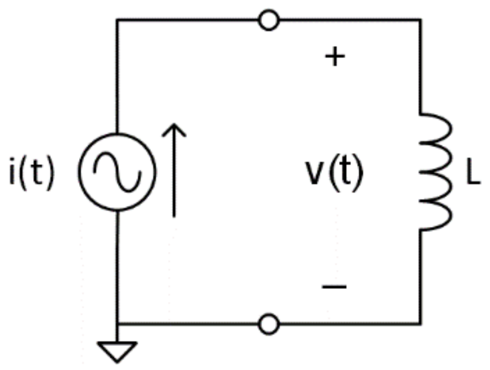
$$\mathbf{I} = \omega C \mathbf{V} e^{j90°}$$

  - Recognizing that $e^{j90°} = j$, we have

$$\boxed{\mathbf{I} = j\omega C\mathbf{V}} \qquad \boxed{\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}}$$

V-I Relationships - Inductor

- ***Inductor***
  - Voltage across an inductor given by



$$v(t) = L\frac{di}{dt}$$

$$v(t) = L\frac{d}{dt}\left[I_p\cos(\omega t + \phi)\right]$$

$$v(t) = -\omega L I_p \sin(\omega t + \phi) = \omega L I_p \cos(\omega t + \phi + 90°)$$

  - Converting to ***phasor form***

$$\mathbf{V} = \omega L I_p e^{j(\phi + 90°)} = \omega L I_p e^{j\phi} e^{j90°}$$

  - Again, recognizing that $e^{j90°} = j$, gives

$$\boxed{\mathbf{V} = j\omega L\mathbf{I}} \qquad \boxed{\mathbf{I} = \frac{1}{j\omega L}\mathbf{V}}$$

Impedance

Impedance

- For resistors, Ohm's law gives the ratio of the ***voltage phasor*** to the ***current phasor*** as

$$\frac{\mathbf{V}}{\mathbf{I}} = R$$

  - R is, of course, ***resistance***
    - A special case of ***impedance***

- ***Impedance***, Z

$$\boxed{Z = \frac{\mathbf{V}}{\mathbf{I}}}$$

  - The ***ratio*** of the ***voltage phasor*** to the ***current phasor*** for a component or network
  - Units: ohms (Ω)
  - In general, complex-valued

Impedance

- ***Resistor impedance***:

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = R$$

- ***Capacitor impedance***:

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- ***Inductor impedance***:

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = j\omega L$$

- In general, ***Ohm's law*** can be applied to any component or network in the ***phasor domain***

$$\mathbf{V} = \mathbf{I}Z$$

$$\mathbf{I} = \frac{\mathbf{V}}{Z}$$

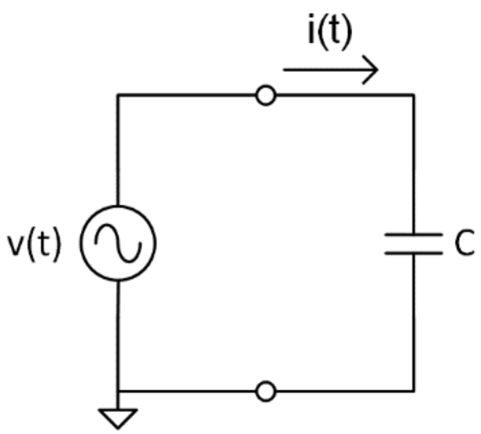
Capacitor Impedance

Capacitor Impedance

$$Z = \frac{1}{j\omega C} = \frac{1}{\omega C} e^{-j90°}$$

$$\mathbf{V} = \mathbf{I}Z = \frac{\mathbf{I}}{\omega C} e^{-j90°}$$

$$\mathbf{I} = \omega C \mathbf{V} e^{j90°}$$



- In the time domain, this translates to

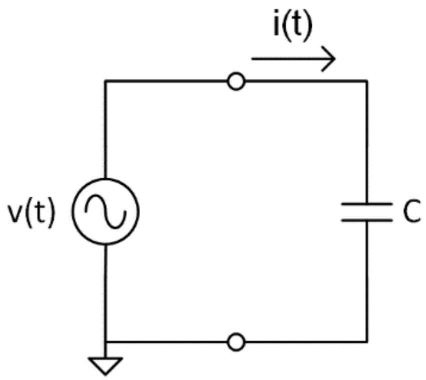
$$v(t) = V_p \cos(\omega t + \phi)$$

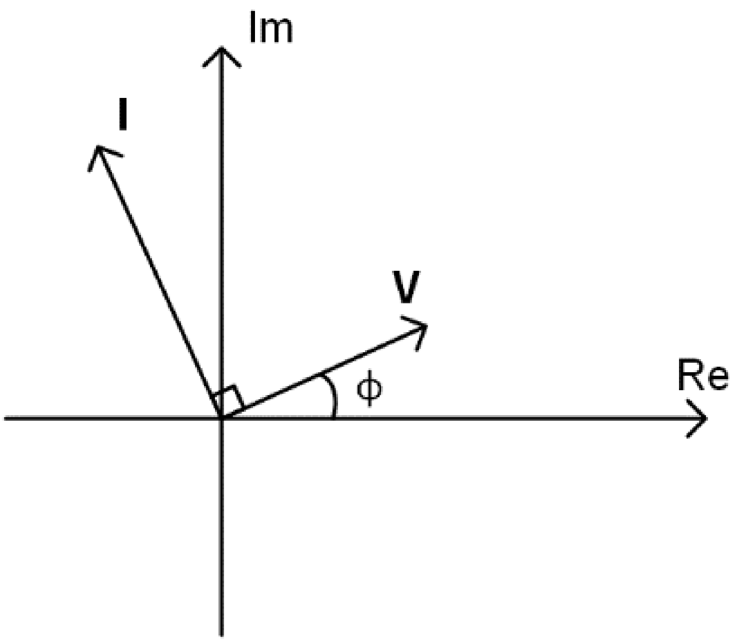
$$i(t) = V_p \omega C \cos(\omega t + \phi + 90°)$$

- ***Current through a capacitor leads the voltage across a capacitor by*** $90°$

Capacitor Impedance – Phasor Diagram

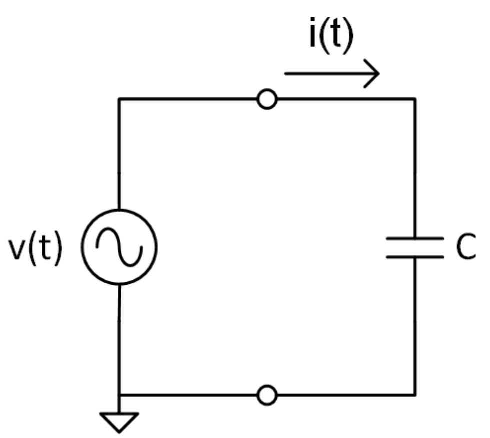
- ***Phasor diagram*** for a capacitor
  - Voltage and current phasors drawn as vectors in the complex plane
  - Current always leads voltage by 90°

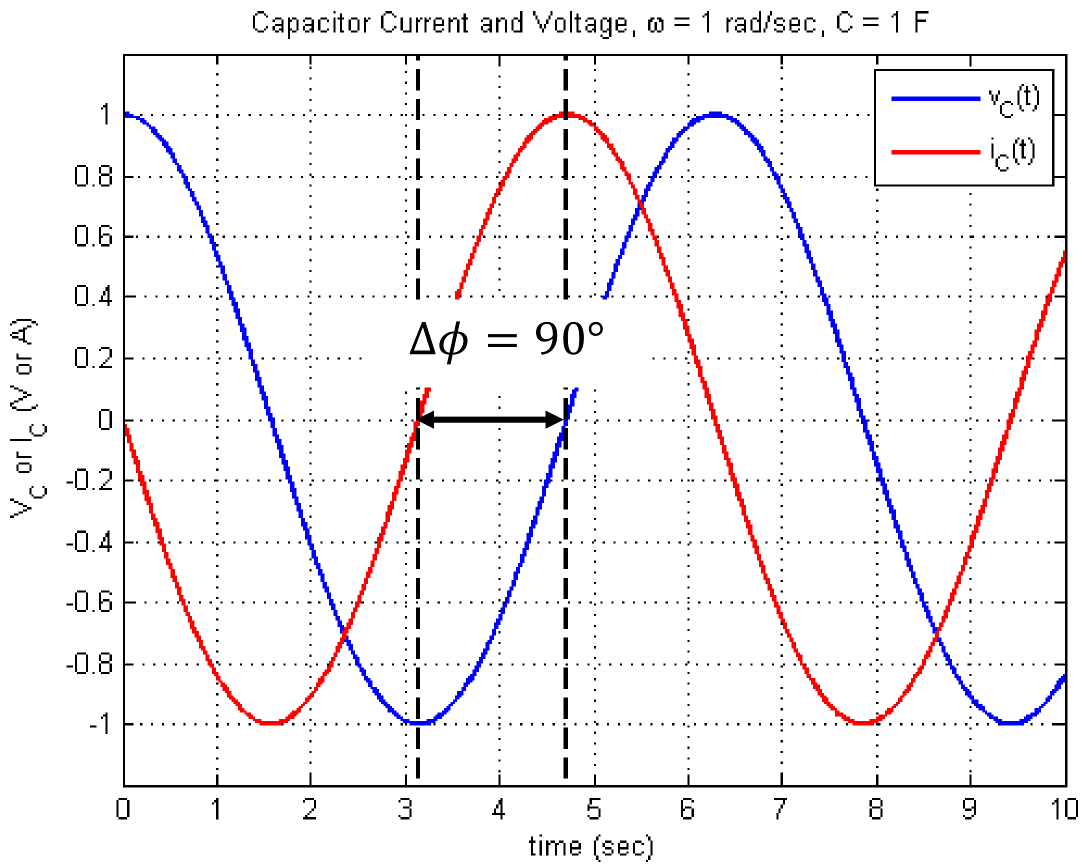




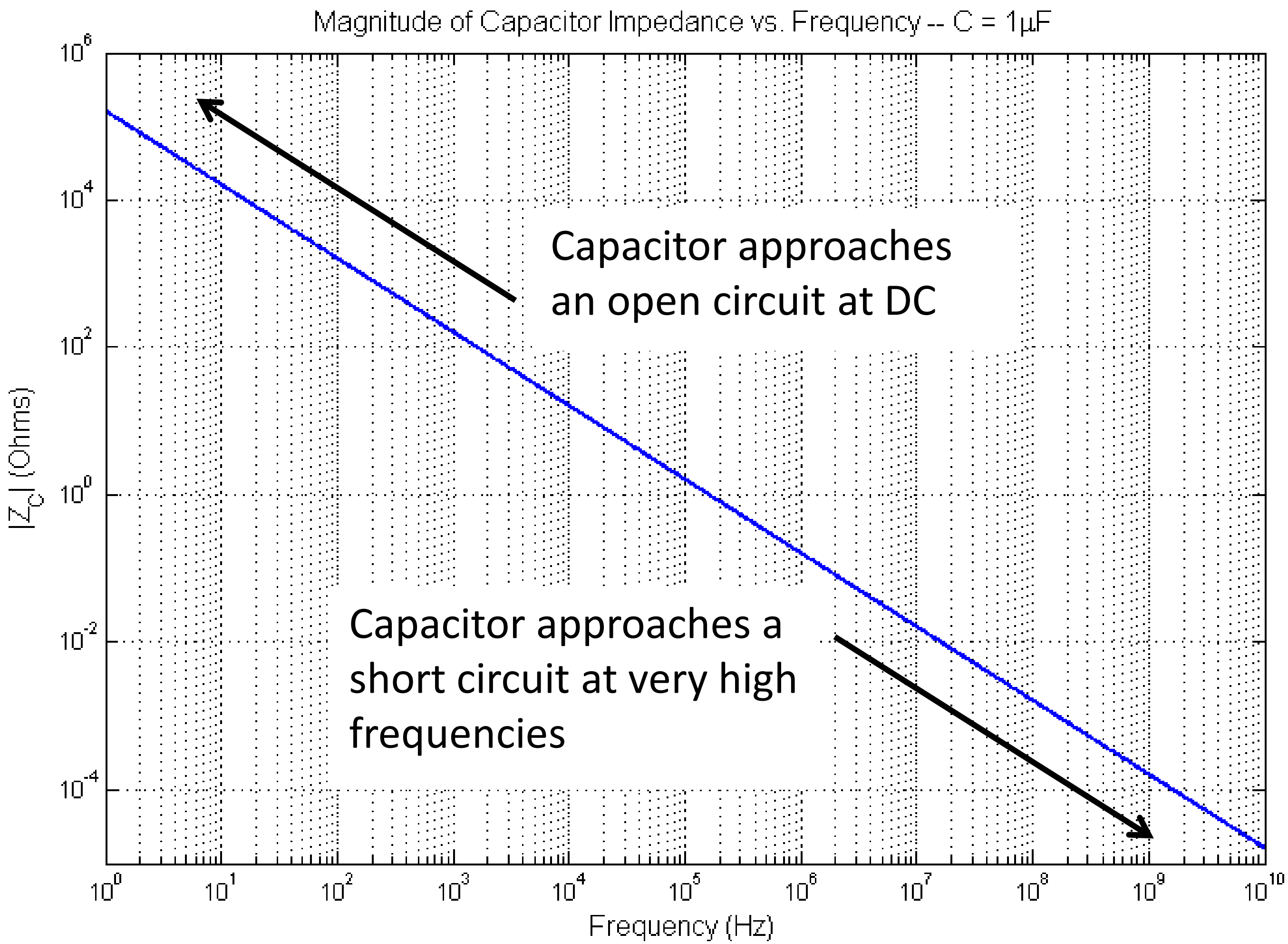
Capacitor Impedance – Time Domain

- Current leads voltage by 90°





Capacitor Impedance – Frequency Domain



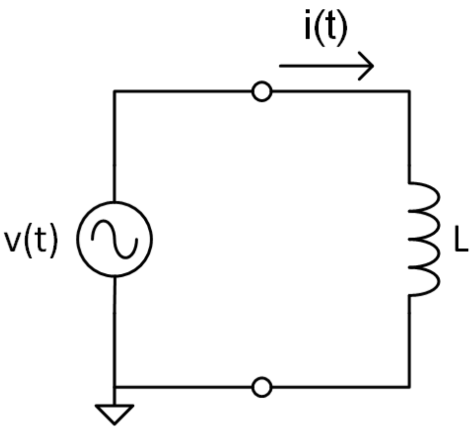
Inductor Impedance

Inductor Impedance

$$Z = j\omega L = \omega L\, e^{j90°}$$

$$\mathbf{V} = \mathbf{I}Z = \mathbf{I}\omega L\, e^{j90°}$$

$$\mathbf{I} = \frac{\mathbf{V}}{\omega L} e^{-j90°}$$



- In the time domain, this translates to

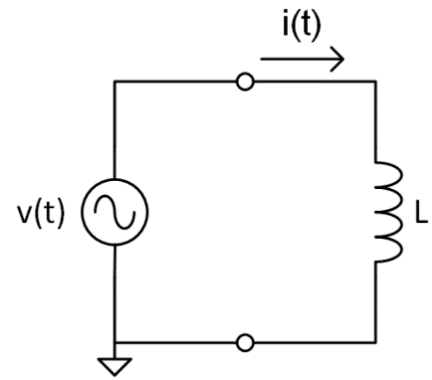
$$v(t) = V_p \cos(\omega t + \phi)$$

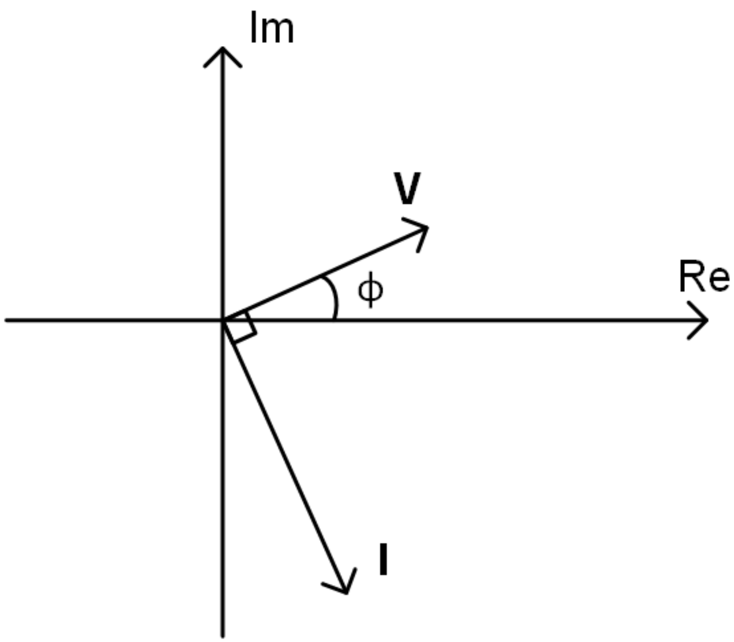
$$i(t) = \frac{V_p}{\omega L} \cos(\omega t + \phi - 90°)$$

- ***Current through an inductor lags the voltage across an inductor by*** $90°$

Inductor Impedance – Phasor Diagram

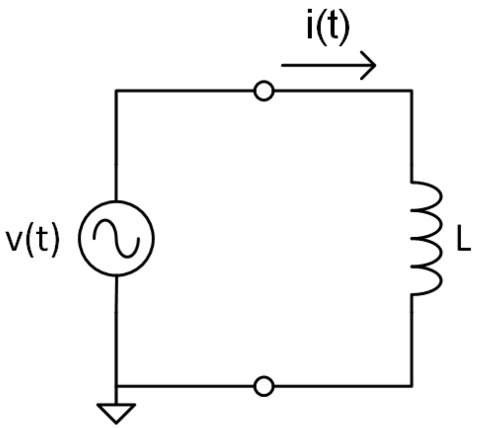
- ***Phasor diagram*** for an inductor
  - Voltage and current phasors drawn as vectors in the complex plane
  - Current always lags voltage by 90°

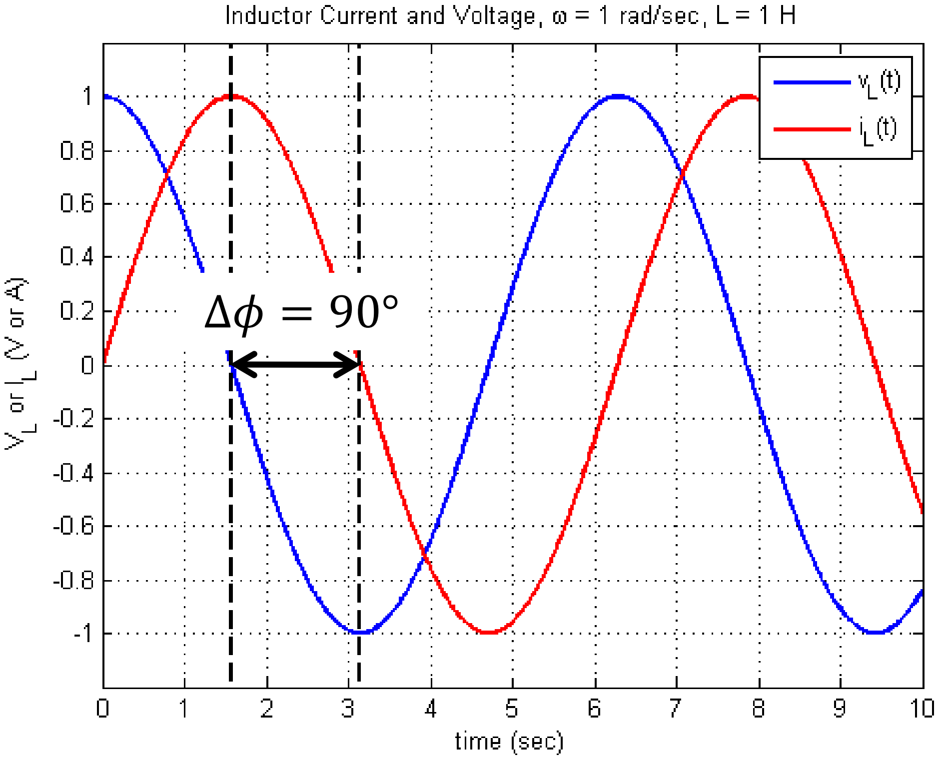




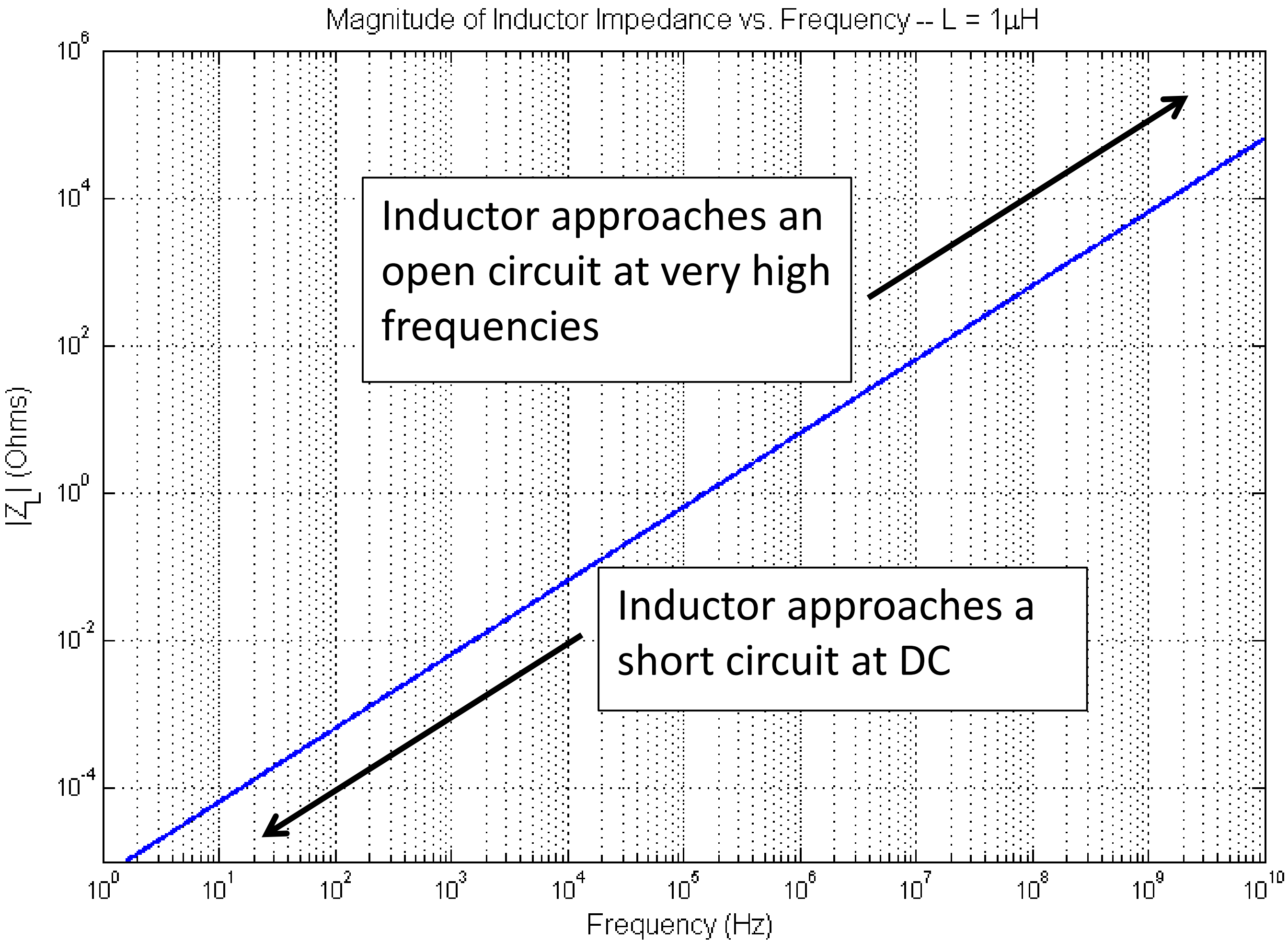
Inductor Impedance – Time Domain

- Current lags voltage by 90°





Inductor Impedance – Frequency Domain



Summary

Capacitor

- Impedance:

$$Z_c = \frac{1}{j\omega C}$$

- V-I phase relationship:

Current leads voltage by 90°

$$v(t) = V_p \cos(\omega t)$$

$$i(t) = V_p \omega C \cos(\omega t + 90°)$$

Inductor

- Impedance:

$$Z_L = j\omega L$$

- V-I phase relationship:

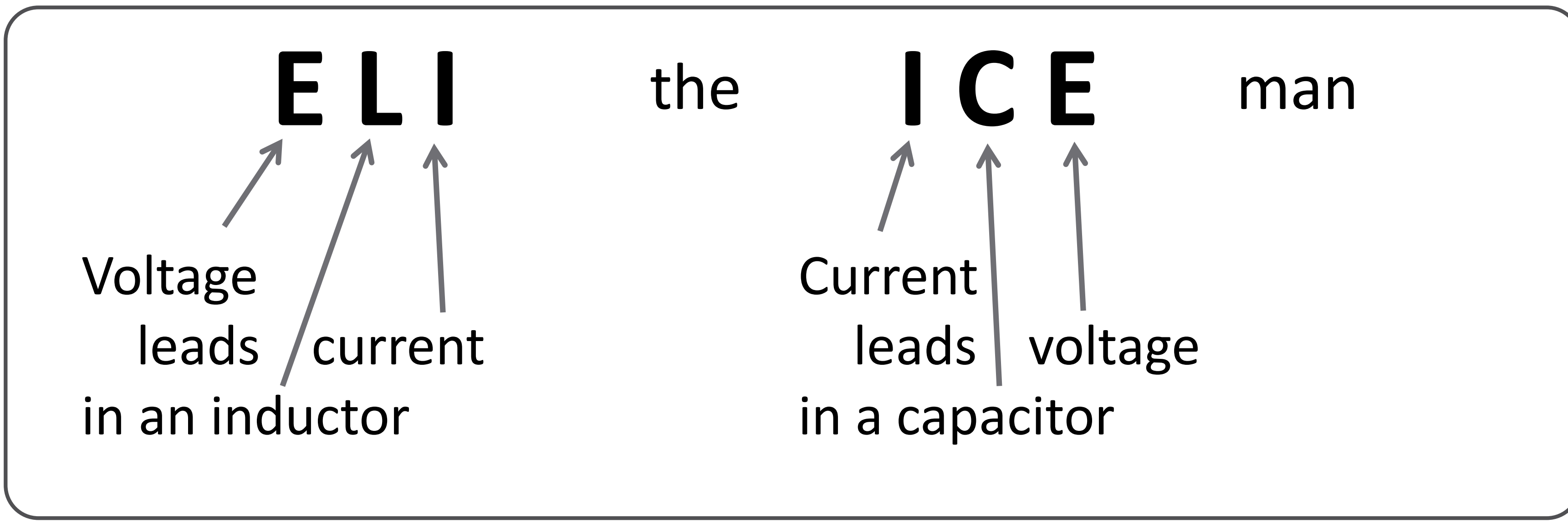
Current lags voltage by 90°

$$v(t) = V_p \cos(\omega t)$$

$$i(t) = \frac{V_p}{\omega L} \cos(\omega t - 90°)$$

ELI the ICE Man

- Mnemonic for phase relation between current (I) and voltage (E) in inductors (L) and capacitors (C)



Impedance of Arbitrary Networks

Impedance

- So far, we've looked at impedance of individual components
  - ***Resistors***

$$Z = R$$

    - Purely real
  - ***Capacitors***

$$Z = \frac{1}{j\omega C}$$

    - Purely imaginary, ***purely reactive***
  - ***Inductors***

$$Z = j\omega L$$

    - Purely imaginary, ***purely reactive***

Impedance

- Also want to be able to characterize the impedance of electrical networks
  - Multiple components
  - Some resistive, some reactive
- In general, impedance is a complex value

$$Z = R + jX$$

where
    - R is ***resistance***
    - X is ***reactance***

- So, in ENGR 201 we dealt with impedance all along
  - ***Resistance*** is an impedance whose reactance (imaginary part) is zero
    - A ***purely real*** impedance

Reactance

- For ***capacitor*** and ***inductors***, impedance is ***purely reactive***
  - Resistive part is zero

$$Z_c = jX_c \quad \text{and} \quad Z_L = jX_L$$

where X_c is ***capacitive reactance***

$$X_c = -\frac{1}{\omega C}$$

and X_L is ***inductive reactance***

$$X_L = \omega L$$

  - Note that ***reactance*** is a ***real*** quantity
    - It is the ***imaginary part*** of impedance
  - Units of reactance: ohms (Ω)

Admittance

- ***Admittance***, Y, is the inverse of impedance

$$\boxed{Y = \frac{1}{Z} = G + jB}$$

where

G is ***conductance*** – the real part

B is ***susceptance*** – the imaginary part

$$Y = \frac{1}{R + jX} = \left(\frac{R}{R^2 + X^2}\right) + j\left(\frac{-X}{R^2 + X^2}\right)$$

- ***Conductance***

$$G = \frac{R}{R^2 + X^2}$$

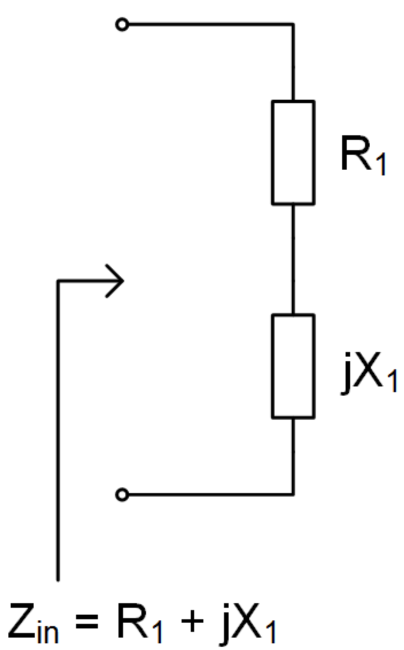
  - Note that $G \neq 1/R$ unless $X = 0$
- ***Susceptance***

$$B = \frac{-X}{R^2 + X^2}$$

- Units of Y, G, and B: Siemens (S)

Impedance of Arbitrary Networks

- In general, the impedance of arbitrary networks may be both resistive and reactive



$$Z = R_1 + jX_1$$

$$Z = |Z|\angle\theta$$

where

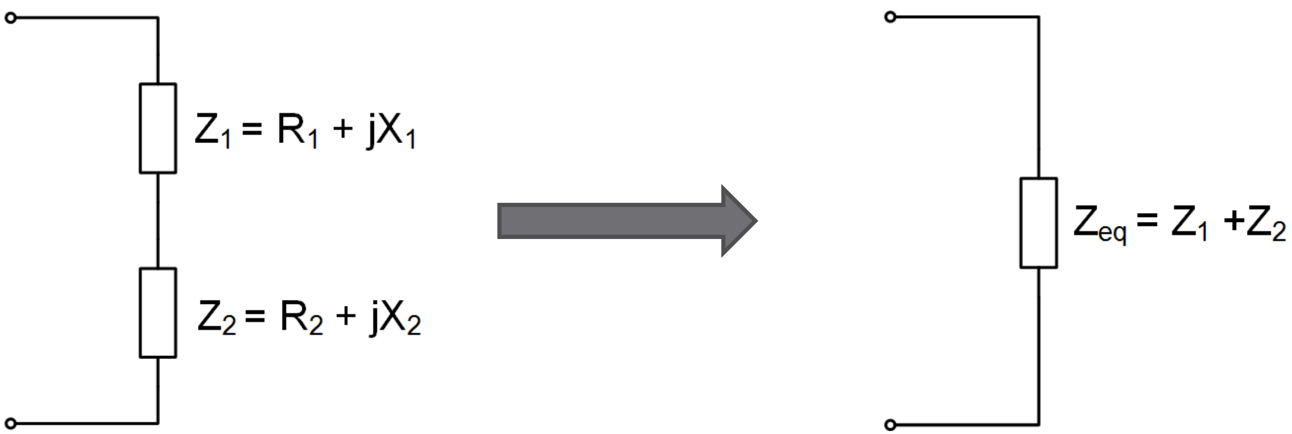
$$|Z| = \sqrt{R_1^2 + X_1^2}$$

and

$$\theta = \tan^{-1}\left(\frac{X_1}{R_1}\right)$$

Impedances in Series

- ***Impedances in series add***

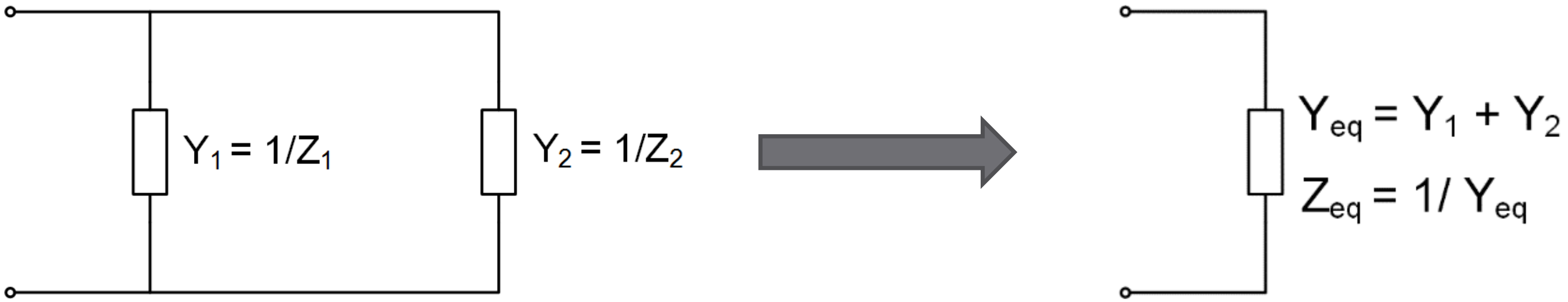


$$Z_{eq} = Z_1 + Z_2$$

$$Z_{eq} = (R_1 + R_2) + j(X_1 + X_2)$$

Impedances in Parallel

- ***Admittances in parallel add***



$$Y_{eq} = Y_1 + Y_2$$

$$Z_{eq} = \frac{1}{Y_{eq}} = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)^{-1}$$

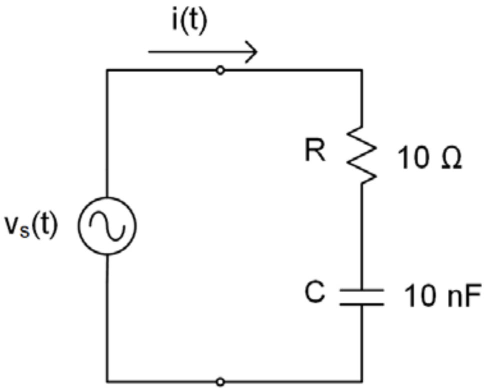
$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Sinusoidal Steady-State Analysis

Sinusoidal Steady-State Analysis – Ex. 1

- Determine the current, $i(t)$

$$v_s(t) = 1\,V\cos(2\pi\cdot 1\,MHz\cdot t)$$



- First, convert the circuit to the ***phasor domain***
  - Express the source voltage as a phasor

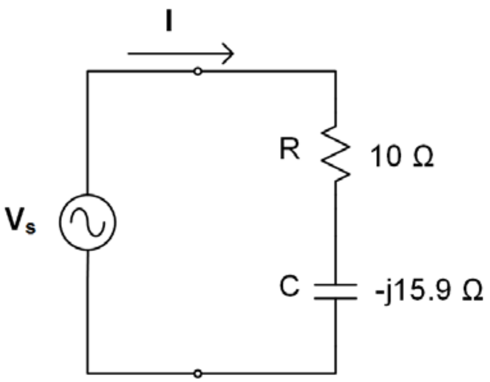
$$\mathbf{V_s} = 1\,V\angle 0°$$

  - Evaluate impedances at 1 MHz

$$R = 10\,\Omega$$

$$Z_c = \frac{1}{j\omega C} = -\frac{j}{2\pi\cdot 1\,MHz\cdot 10\,nF}$$

$$Z_c = -j15.9\,\Omega$$



Sinusoidal Steady-State Analysis – Ex. 1

- The load impedance is

$$Z = R + jX_c = (10 - j15.9)\ \Omega$$

$$Z = 18.8\angle -57.8°\ \Omega$$

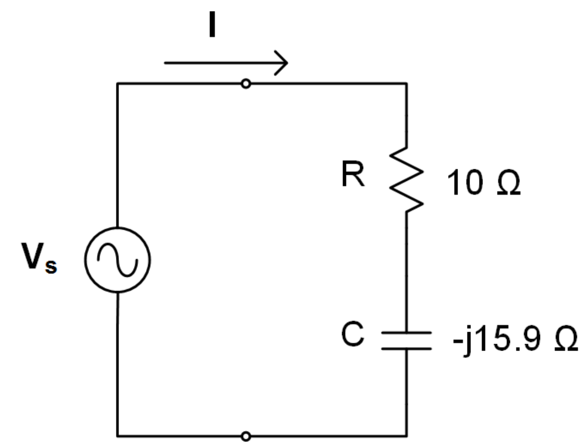
- Apply Ohm's law to calculate the current phasor

$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{1\ V\angle 0°}{18.8\angle -57.8°\ \Omega}$$

$$\mathbf{I} = 53.2\angle 57.8°\ mA$$

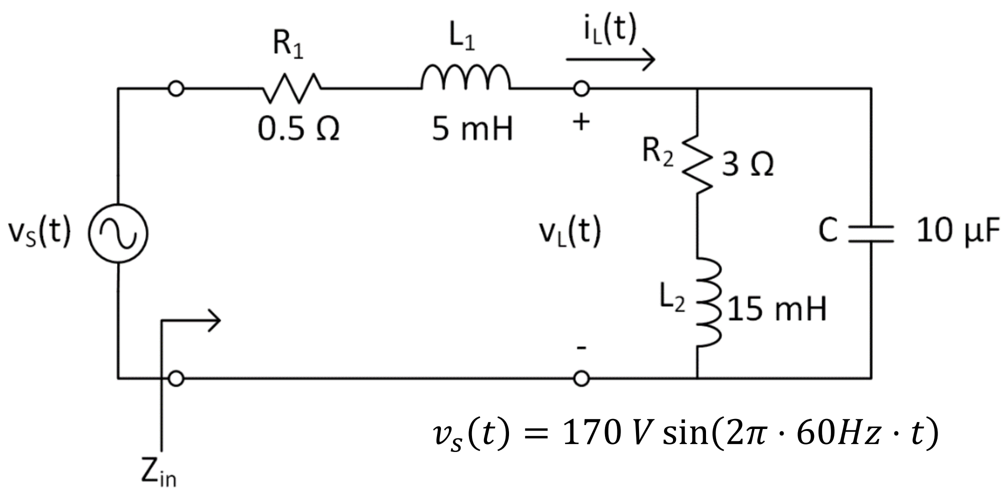
- Finally, convert to the ***time domain***

$$i(t) = 53.2\ mA\cos(2\pi \cdot 1MHz \cdot t + 57.8°)$$



Sinusoidal Steady-State Analysis – Ex. 2

- Consider the following circuit, modeling a source driving a load through a transmission line

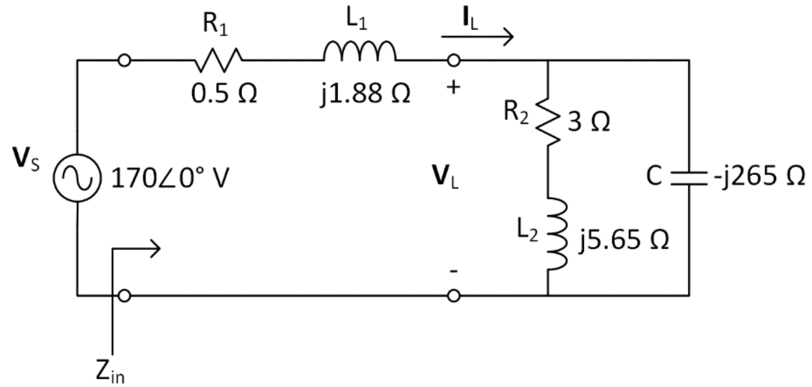


$v_s(t) = 170\,V\,\sin(2\pi \cdot 60Hz \cdot t)$

- Determine:
  - The impedance, Z_{in}, at 60 Hz
  - Voltage across the load, $v_L(t)$
  - Current delivered to the load, $i_L(t)$

Sinusoidal Steady-State Analysis – Ex. 2

- First, convert to the phasor domain and evaluate impedances at 60 Hz



- The line impedance is

$$Z_{line} = R_1 + j\omega L_1 = 0.5 + j1.88\ \Omega$$

- The load impedance is

$$Z_{load} = (R_2 + j\omega L_2) || \frac{1}{j\omega C} = (3 + j5.65\ \Omega) || -j265\ \Omega$$

$$Z_{load} = \left(\frac{1}{3 + j5.65\ \Omega} + \frac{1}{-j265\ \Omega} \right)^{-1} = 3.13 + j5.74\ \Omega$$

Sinusoidal Steady-State Analysis – Ex. 2

- The impedance seen by the source is

$$Z_{in} = Z_{line} + Z_{load}$$

$$Z_{in} = (0.5 + j1.88\ \Omega) + (3.13 + j5.74\ \Omega)$$

$$Z_{in} = 3.63 + j7.62\ \Omega$$

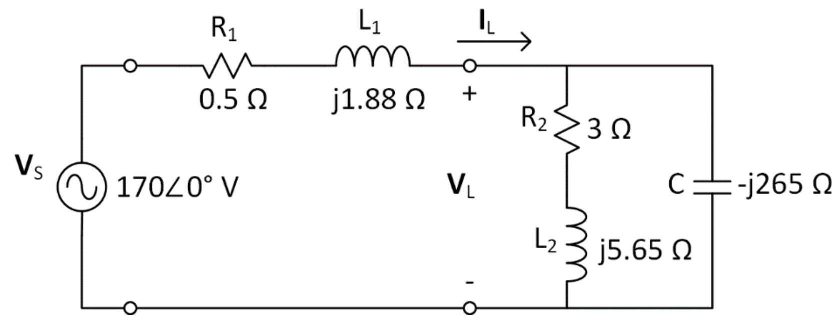
- In polar form:

$$Z_{in} = 8.44\angle 64.5°\ \Omega$$

- The impedance driven by the source looks resistive and inductive
  - Resistive: non-zero resistance, $\angle Z_{in} \neq \pm 90°$
  - Inductive: positive reactance, positive angle

Sinusoidal Steady-State Analysis – Ex. 2

- Apply ***voltage division*** to determine the voltage across the load



$$\mathbf{V}_L = \mathbf{V}_S \frac{Z_{load}}{Z_{line} + Z_{load}}$$

$$\mathbf{V}_L = 170\angle 0°\,V \frac{3.13 + j5.74\,\Omega}{3.63 + j7.62\,\Omega}$$

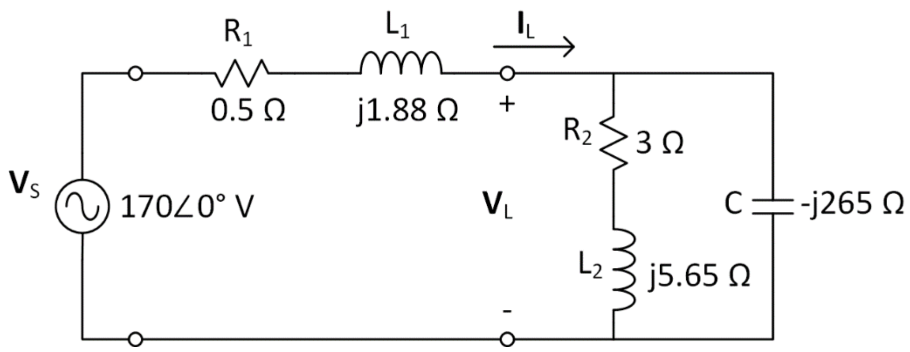
$$\mathbf{V}_L = 170\angle 0°\,V \frac{6.54\angle 61.4°\,\Omega}{8.44\angle 64.5°\,\Omega} = 132\angle -3.1°\,V$$

- Converting to ***time-domain*** form

$$v_L(t) = 132\,V\sin(2\pi \cdot 60Hz \cdot t - 3.1°)$$

Sinusoidal Steady-State Analysis – Ex. 2

- Finally, calculate the current delivered to the load



$$\mathbf{I}_L = \frac{\mathbf{V}_L}{Z_{load}}$$

$$\mathbf{I}_L = \frac{132\angle -3.1°\ V}{6.54\angle 61.4°\ \Omega}$$

$$\mathbf{I}_L = 20.1\angle -64.5°\ A$$

- In time-domain form:

$$i_L(t) = 20.1\ A \sin(2\pi \cdot 60Hz \cdot t - 64.5°)$$