SECTION 1: SINUSOIDAL STEADY-STATE ANALYSIS
Sinusoids
Sinusoidal Signals

- **Sinusoidal** signals are of particular interest in the field of electrical engineering.

\[ v(t) = V_p \cos(\omega t + \phi) = V_p \cos(2\pi f t + \phi) \]

- Sinusoidal signals defined by three parameters:
  - **Amplitude**: \( V_p \)
  - **Frequency**: \( \omega \) or \( f \)
  - **Phase**: \( \phi \)
- **Amplitude** of a sinusoid is its *peak* voltage, $V_p$

- **Peak-to-peak voltage**, $V_{pp}$, is twice the amplitude
  - $V_{pp} = 2V_p$
  - $V_{pp} = V_{max} - V_{min}$

Where $v(t) = V_p \cdot \sin(\omega t + \phi) = V_p \cdot \sin(2\pi f t + \phi)$

- $V_p = 170 \text{ V}$
- $V_{pp} = 340 \text{ V}$
Frequency

- **Period** ($T$)
  - Duration of one cycle

- **Frequency** ($f$)
  - Number of periods per second
    \[ f = \frac{1}{T} \]

- **Ordinary frequency**, $f$
  - Units: hertz (Hz), sec$^{-1}$, cycles/sec

- **Angular frequency**, $\omega$
  - Units: rad/sec
    \[ \omega = 2\pi f, \quad f = \frac{\omega}{2\pi} \]

\[ T = 16 \text{ msec} \]
Phase

Angular constant in signal expression, $\phi$

$$v(t) = V_p \sin(\omega t + \phi)$$

Requires a time reference
- Interested in relative, not absolute, phase

Here,
- $v_1(t)$ leads $v_2(t)$
- $v_2(t)$ lags $v_1(t)$

Units: radians
- Not technically correct, but OK to express in degrees, e.g.:

$$v(t) = 170 \ V \sin(2\pi \cdot 60Hz \cdot t + 34^\circ)$$
Sinusoidal Steady-State Analysis

- Often interested in the response of linear systems to *sinusoidal inputs*
  - Voltages and currents in electrical systems
  - Forces, torques, velocities, etc. in mechanical systems
- For *linear systems* excited by a sinusoidal input
  - Output is sinusoidal
    - *Same frequency*
    - In general, *different amplitude*
    - In general, *different phase*

- We can simplify the analysis of linear systems by using *phasor representation* of sinusoids
Phasors

- **Phasor**
  - A *complex number* representing the *amplitude* and *phase* of a sinusoidal signal
  - Frequency is not included
    - Remains constant and is accounted for separately
    - System characteristics (frequency-dependent) evaluated at the frequency of interest as first step in the analysis

- Phasors are *complex numbers*
  - Before applying phasors to the analysis of electrical circuits, we’ll first review the properties of complex numbers
Complex Numbers
Complex Numbers

- A complex number can be represented as
  \[ z = x + jy \]
  - \( x \): real part (a real number)
  - \( y \): imaginary part (a real number)
  - \( j = \sqrt{-1} \) is the imaginary unit

- Complex numbers can be represented three ways:
  - **Cartesian** form: \( z = x + jy \)
  - **Polar** form: \( z = r\angle \phi \)
  - **Exponential** form: \( z = re^{j\phi} \)
Complex Numbers as Vectors

- A complex number can be represented as a *vector in the complex plane*

- Complex plane
  - *Real axis* – horizontal
  - *Imaginary axis* – vertical

- A vector from the origin to \( z \)
  - Real part, \( x \)
  - Imaginary part, \( y \)

\[
z = x + jy
\]

- Vector has a *magnitude*, \( r \)
- And an *angle*, \( \theta \)

\[
z = r \angle \theta
\]
Cartesian Form ↔ Polar Form

- Cartesian form → Polar form
  
  \[ z = x + jy = r \angle \theta \]
  
  \[ r = |z| = \sqrt{x^2 + y^2} \]
  
  \[ \theta = \text{arg}(z) = \angle z \]
  
  \[ \theta = \tan^{-1}\left(\frac{y}{x}\right) \]

- Polar form → Cartesian form
  
  \[ x = r \cos(\theta) \]
  
  \[ y = r \sin(\theta) \]
Complex Numbers – Addition/Subtraction

- **Addition and subtraction** of complex numbers
  - Best done in *Cartesian* form
  - Real parts add/subtract
  - Imaginary parts add/subtract

- For example:

  \[
  z_1 = x_1 + jy_1 \\
  z_2 = x_2 + jy_2 \\
  z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \\
  z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)
  \]
Complex Numbers – Multiplication/Division

- **Multiplication and division** of complex numbers
  - Best done in *polar* form
  - Magnitudes multiply/divide
  - Angles add/subtract

- For example:

  \[
  z_1 = r_1 \angle \theta_1 \\
  z_2 = r_2 \angle \theta_2 \\
  z_1 \cdot z_2 = r_1 r_2 \angle (\theta_1 + \theta_2) \\
  \frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2)
  \]
Complex Conjugate

- **Conjugate** of a complex number
  - Number that results from *negating the imaginary part*
    \[ z = x + jy \]
    \[ z^* = x - jy \]
  - Or, equivalently, from *negating the angle*
    \[ z = r \angle \theta \]
    \[ z^* = r \angle -\theta \]
Complex Fractions

- Multiplying a number by its complex conjugate yields the squared magnitude of that number
  - A real number
    
    \[ z \cdot z^* = (x + jy)(x - jy) = x^2 + y^2 \]
    
    \[ z \cdot z^* = r \angle \theta \cdot r \angle -\theta = r^2 \angle \theta - \theta = r^2 \]

- Rationalizing the denominator of a complex fraction:
  - Multiply numerator and denominator by the complex conjugate of the denominator
    
    \[ z = \frac{x_1 + jy_1}{x_2 + jy_2} \cdot \frac{x_2 - jy_2}{x_2 - jy_2} \]
    
    \[ z = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j \frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \]
Complex Fractions

- **Fractions** or **ratios** are, of course, simply division
  - Very common form, so worth emphasizing
- **Magnitude** of a ratio of complex numbers
  \[ z = \frac{Z_1}{Z_2} \quad \rightarrow \quad |z| = \frac{|Z_1|}{|Z_2|} \]
- **Angle** of a ratio of complex numbers
  \[ z = \frac{Z_1}{Z_2} \quad \rightarrow \quad \angle z = \angle Z_1 - \angle Z_2 \]

- **Calculators and complex numbers**
  - Manipulation of complex numbers by hand is tedious and error-prone
  - Your calculators can perform complex arithmetic
  - They will operate in both Cartesian and polar form, and will convert between the two
  - Learn to use them – correctly
Phasors
Euler’s Identity

- Fundamental to phasor notation is Euler’s identity:

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]

where \( j \) is the imaginary unit, and \( \omega \) is angular frequency.

- It follows that

\[
\cos(\omega t) = Re\{e^{j\omega t}\} \\
\sin(\omega t) = Im\{e^{j\omega t}\}
\]

and

\[
\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \\
\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}
\]
Phasors

□ Consider a sinusoidal voltage

\[ v(t) = V_p \cos(\omega t + \phi) \]

□ Using Euler’s identity, we can represent this as

\[ v(t) = Re\{V_p e^{j(\omega t + \phi)}\} = Re\{V_p e^{j\phi} e^{j\omega t}\} \]

where

- \( V_p \) represents magnitude
- \( e^{j\phi} \) represents phase
- \( e^{j\omega t} \) represents a sinusoid of frequency \( \omega \)

□ Grouping the first two terms together, we have

\[ v(t) = Re\{Ve^{j\omega t}\} \]

where \( V \) is the phasor representation of \( v(t) \)
Phasors

\[ v(t) = Re \{ V e^{j\omega t} \} \]

- The phasor representation of \( v(t) \)
  \[ V = V_p e^{j\phi} \]

- A representation of magnitude and phase only
- Time-harmonic portion \( (e^{j\omega t}) \) has been dropped

<table>
<thead>
<tr>
<th>Time-domain representation:</th>
<th>Phasor-domain representation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(t) = V_p \sin(\omega t + \phi) )</td>
<td>( V = V_p e^{j\phi} = V_p \angle \phi )</td>
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- Phasors greatly simplify sinusoidal steady-state analysis
  - Messy trigonometric functions are eliminated
  - Differentiation and integration transformed to algebraic operations
Voltage & Current in the Phasor Domain

- We will use phasors to simplify analysis of electrical circuits
  - Need an understanding of electrical component behavior in the phasor domain
  - Relationships between *voltage phasors* and *current phasors* for Rs, Ls, and Cs

- **Resistor**
  - Voltage across a resistor given by
    \[ v(t) = i(t)R \]
    \[ i(t) = I_p \cos(\omega t + \phi) \]
  - Converting to phasor form
    \[ V = (I_p e^{j\phi})R \]
    \[ V = IR \quad I = \frac{V}{R} \]
  - *Ohm’s law* in phasor form
**V-I Relationships in the Phasor Domain**

- **Capacitor**
  - Current through the capacitor given by
  
  \[ i(t) = C \frac{dv}{dt} \]

  \[ i(t) = C \frac{d}{dt} [V_p \cos(\omega t + \phi)] \]

  \[ i(t) = -\omega CV_p \sin(\omega t + \phi) \]

  - Applying a trig identity:
    
    \[ -\sin(A) = \cos(A + 90^\circ) \]

    gives

    \[ i(t) = \omega CV_p \cos(\omega t + \phi + 90^\circ) \]

  - Converting to **phasor form**

    \[ I = \omega CV_p e^{j(\phi + 90^\circ)} = \omega CV_p e^{j\phi} e^{j90^\circ} \]
V-I Relationships - Capacitor

- **Current phasor**
  \[ I = \omega CV_p e^{j(\phi + 90^\circ)} = \omega CV_p e^{j\phi} e^{j90^\circ} \]

- **Voltage phasor** is
  \[ V = V_p e^{j\phi} \]
  so
  \[ I = \omega CV e^{j90^\circ} \]
  Recognizing that \( e^{j90^\circ} = j \), we have
  \[ I = j\omega CV \]
  \[ V = \frac{1}{j\omega C} I \]
**V-I Relationships - Inductor**

- **Inductor**
  - Voltage across an inductor given by
  
  \[ v(t) = L \frac{di}{dt} \]

  \[ v(t) = L \frac{d}{dt}[I_p \cos(\omega t + \phi)] \]

  \[ v(t) = -\omega LI_p \sin(\omega t + \phi) = \omega LI_p \cos(\omega t + \phi + 90^\circ) \]

- Converting to **phasor form**
  
  \[ \mathbf{V} = \omega LI_p e^{j(\phi+90^\circ)} = \omega LI_p e^{j\phi} e^{j90^\circ} \]

- Again, recognizing that \( e^{j90^\circ} = j \), gives

\[
\begin{align*}
\mathbf{V} &= j\omega LI \\
\mathbf{I} &= \frac{1}{j\omega L} \mathbf{V}
\end{align*}
\]
Impedance
Impedance

- For resistors, Ohm’s law gives the ratio of the voltage phasor to the current phasor as
  \[ \frac{V}{I} = R \]

  - \( R \) is, of course, resistance
    - A special case of impedance

- **Impedance, \( Z \)**
  \[ Z = \frac{V}{I} \]

  - The ratio of the voltage phasor to the current phasor for a component or network
  - Units: ohms (\( \Omega \))
  - In general, complex-valued
Impedance

- **Resistor impedance:**
  \[ Z = \frac{V}{I} = R \]

- **Capacitor impedance:**
  \[ Z = \frac{V}{I} = \frac{1}{j\omega C} \]

- **Inductor impedance:**
  \[ Z = \frac{V}{I} = j\omega L \]

- In general, **Ohm’s law** can be applied to any component or network in the **phasor domain**
  \[ V = IZ \]
  \[ I = \frac{V}{Z} \]
Capacitor Impedance
Capacitor Impedance

\[ Z = \frac{1}{j\omega C} = \frac{1}{\omega C} e^{-j90^\circ} \]

\[ V = IZ = \frac{I}{\omega C} e^{-j90^\circ} \]

\[ I = \omega CV e^{j90^\circ} \]

- In the time domain, this translates to
  \[ v(t) = V_p \cos(\omega t + \phi) \]
  \[ i(t) = V_p \omega C \cos(\omega t + \phi + 90^\circ) \]

- **Current through a capacitor leads the voltage across a capacitor by 90°**
**Phasor diagram** for a capacitor

- Voltage and current phasors drawn as vectors in the complex plane
- Current always leads voltage by 90°
Current leads voltage by 90°

\[ \Delta \phi = 90° \]
Capacitor Impedance – Frequency Domain

- Capacitor approaches an open circuit at DC.
- Capacitor approaches a short circuit at very high frequencies.

Magnitude of Capacitor Impedance vs. Frequency -- C = 1μF
Inductor Impedance
Inductor Impedance

\[ Z = j\omega L = \omega L e^{j90^\circ} \]

\[ V = IZ = I\omega L e^{j90^\circ} \]

\[ I = \frac{V}{\omega L} e^{-j90^\circ} \]

- In the time domain, this translates to

\[ v(t) = V_p \cos(\omega t + \phi) \]

\[ i(t) = \frac{V_p}{\omega L} \cos(\omega t + \phi - 90^\circ) \]

- Current through an inductor lags the voltage across an inductor by 90°
**Phasor diagram** for an inductor

- Voltage and current phasors drawn as vectors in the complex plane
- Current always lags voltage by 90°
Current lags voltage by 90°

\[ \Delta \phi = 90° \]
Inductor Impedance – Frequency Domain

Inductor approaches a short circuit at DC

Inductor approaches an open circuit at very high frequencies
Summary

**Capacitor**
- Impedance:
  \[ Z_C = \frac{1}{j\omega C} \]
- V-I phase relationship:
  Current leads voltage by 90°
  \[ v(t) = V_p \cos(\omega t) \]
  \[ i(t) = V_p \omega C \cos(\omega t + 90°) \]

**Inductor**
- Impedance:
  \[ Z_L = j\omega L \]
- V-I phase relationship:
  Current lags voltage by 90°
  \[ v(t) = V_p \cos(\omega t) \]
  \[ i(t) = \frac{V_p}{\omega L} \cos(\omega t - 90°) \]
ELI the ICE Man

- Mnemonic for phase relation between current (I) and voltage (E) in inductors (L) and capacitors (C)

**ELI**
- Voltage leads current in an inductor

**the**

**ICE**
- Current leads voltage in a capacitor

**ELI**
- Man
Example Problems
Convert each of the following time-domain signals to phasor form.

\[ v(t) = 6V \cdot \cos(2\pi \cdot 8kHz \cdot t + 12^\circ) \]
\[ i(t) = 200mA \cdot \sin(100 \cdot t - 38^\circ) \]
Convert the following circuit to the phasor domain.
The following current is applied to the capacitor.

\[ i(t) = 100mA \cdot \cos(2\pi \cdot 50kHz \cdot t) \]

Find the voltage across the capacitor, \( v(t) \).
\[ i(t) \uparrow \quad v(t) \quad 0.1 \mu F \]
The following voltage is applied to the inductor.

\[ v(t) = 4V \cdot \cos(2\pi \cdot 800Hz \cdot t) \]

Find the current through the inductor, \( i(t) \).
A test voltage is applied to the input of an electrical network.

\[ v(t) = 1V \cdot \sin(2\pi \cdot 5kHz \cdot t) \]

The input current is measured.

\[ i(t) = 268mA \cdot \sin(2\pi \cdot 5kHz \cdot t - 46^\circ) \]

What is the circuit’s input impedance, \( Z_{in} \)?
Impedance of Arbitrary Networks
Impedance

- So far, we’ve looked at impedance of individual components

  - **Resistors**
    \[ Z = R \]
    - Purely real

  - **Capacitors**
    \[ Z = \frac{1}{j\omega C} \]
    - Purely imaginary, *purely reactive*

  - **Inductors**
    \[ Z = j\omega L \]
    - Purely imaginary, *purely reactive*
Impedance

- Also want to be able to characterize the impedance of electrical networks
  - Multiple components
  - Some resistive, some reactive

- In general, impedance is a complex value
  \[ Z = R + jX \]

where

- \( R \) is *resistance*
- \( X \) is *reactance*

- So, in ENGR 201 we dealt with impedance all along
  - *Resistance* is an impedance whose reactance (imaginary part) is zero
    - A *purely real* impedance
Reactance

- For *capacitor* and *inductors*, impedance is **purely reactive**
  - Resistive part is zero
    
    \[ Z_c = jX_c \quad \text{and} \quad Z_L = jX_L \]

    where \( X_c \) is *capacitive reactance*
    
    \[ X_c = -\frac{1}{\omega C} \]

    and \( X_L \) is *inductive reactance*
    
    \[ X_L = \omega L \]

- Note that *reactance* is a **real** quantity
  - It is the *imaginary part* of impedance

- Units of reactance: ohms (Ω)
Admittance

- **Admittance**, $Y$, is the inverse of impedance
  
  $$Y = \frac{1}{Z} = G + jB$$

  where
  
  $G$ is **conductance** – the real part
  
  $B$ is **susceptance** – the imaginary part

- **Conductance**
  
  $$G = \frac{R}{R^2 + X^2}$$

  Note that $G \neq 1/R$ unless $X = 0$

- **Susceptance**
  
  $$B = \frac{-X}{R^2 + X^2}$$

- Units of $Y$, $G$, and $B$: Siemens (S)
Impedance of Arbitrary Networks

In general, the impedance of arbitrary networks may be both resistive and reactive

\[ Z = R_1 + jX_1 \]

where

\[ |Z| = \sqrt{R_1^2 + X_1^2} \]

and

\[ \theta = \tan^{-1}\left(\frac{X_1}{R_1}\right) \]
Impedances in Series

- Impedances in series add

\[
Z_{eq} = Z_1 + Z_2
\]

\[
Z_{eq} = (R_1 + R_2) + j(X_1 + X_2)
\]
**Impedances in Parallel**

- **Admittances in parallel add**

\[ Y_{eq} = Y_1 + Y_2 \]

\[ Z_{eq} = \frac{1}{Y_{eq}} = \left( \frac{1}{Z_1} + \frac{1}{Z_2} \right)^{-1} \]

\[ Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2} \]
Sinusoidal Steady-State Analysis
Sinusoidal Steady-State Analysis – Ex. 1

- Determine the current, \( i(t) \)
  \[ v_s(t) = 1 \, V \cos(2\pi \cdot 1 \, MHz \cdot t) \]

- First, convert the circuit to the **phasor domain**
  - Express the source voltage as a phasor
    \[ V_s = 1 \, V \angle 0^\circ \]
  - Evaluate impedances at 1 MHz
    \[ R = 10 \, \Omega \]
    \[ Z_C = \frac{1}{j\omega C} = -\frac{j}{2\pi \cdot 1 \, MHz \cdot 10 \, nF} \]
    \[ Z_C = -j15.9 \, \Omega \]
The load impedance is

\[ Z = R + jX_C = (10 - j15.9) \, \Omega \]

\[ Z = 18.8 \angle -57.8^\circ \, \Omega \]

Apply Ohm’s law to calculate the current phasor

\[ I = \frac{V}{Z} = \frac{1 \, V \angle 0^\circ}{18.8 \angle -57.8^\circ \, \Omega} \]

\[ I = 53.2 \angle 57.8^\circ \, mA \]

Finally, convert to the **time domain**

\[ i(t) = 53.2 \, mA \cos(2\pi \cdot 1MHz \cdot t + 57.8^\circ) \]
Consider the following circuit, modeling a source driving a load through a cable.

\[ v_s(t) = 170 \, V \sin(2\pi \cdot 60\text{Hz} \cdot t) \]

Determine:
- The impedance, \( Z_{in} \), at 60 Hz
- Voltage across the load, \( v_L(t) \)
- Current delivered to the load, \( i_L(t) \)
First, convert to the phasor domain and evaluate impedances at 60 Hz

The line impedance is

\[ Z_{\text{line}} = R_1 + j\omega L_1 = 0.5 + j1.88 \, \Omega \]

The load impedance is

\[ Z_{\text{load}} = (R_2 + j\omega L_2)\left|\frac{1}{j\omega C}\right| = (3 + j5.65 \, \Omega)\left|\frac{1}{-j265 \, \Omega}\right| = 3.13 + j5.74 \, \Omega \]
The impedance seen by the source is

\[ Z_{in} = Z_{line} + Z_{load} \]
\[ Z_{in} = (0.5 + j1.88 \, \Omega) + (3.13 + j5.74 \, \Omega) \]
\[ Z_{in} = 3.63 + j7.62 \, \Omega \]

In polar form:

\[ Z_{in} = 8.44 \angle 64.5^\circ \, \Omega \]

The impedance driven by the source looks resistive and inductive

- Resistive: non-zero resistance, \( \angle Z_{in} \neq \pm 90^\circ \)
- Inductive: positive reactance, positive angle
Apply **voltage division** to determine the voltage across the load

\[ V_L = V_S \frac{Z_{load}}{Z_{line} + Z_{load}} \]

\[ V_L = 170\angle0^\circ V \frac{3.13 + j5.74 \Omega}{3.63 + j7.62 \Omega} \]

\[ V_L = 170\angle0^\circ V \frac{6.54\angle61.4^\circ \Omega}{8.44\angle64.5^\circ \Omega} = 132\angle -3.1^\circ V \]

Converting to **time-domain** form

\[ v_L(t) = 132 V \sin(2\pi \cdot 60Hz \cdot t - 3.1^\circ) \]
Finally, calculate the current delivered to the load

\[ I_L = \frac{V_L}{Z_{load}} \]

\[ I_L = \frac{132\angle - 3.1^\circ}{6.54\angle 61.4^\circ \, \Omega} \]

\[ I_L = 20.1\angle - 64.5^\circ \, A \]

In time-domain form:

\[ i_L(t) = 20.1 \, A \sin(2\pi \cdot 60\, Hz \cdot t - 64.5^\circ) \]
Example Problems
Determine the input impedance and an equivalent circuit model for the following network at 50 kHz.