SECTION 2: FIRST-ORDER FILTERS
Introduction
Filters

- We are all familiar with **water** and **air filters**
  - Basis for operation is **size selectivity**
    - Small particles (e.g. air or water molecules) are allowed to pass
    - Larger particles (e.g. dust, sediment) are not
  - Unwanted components are **filtered out** of the flow.

- **Electrical filters** are similar
  - Basis for operation is **frequency selectivity**
    - Signal components in certain frequency ranges are **filtered out**
    - Signal components at other frequencies are allowed to pass
Noise

- All real-world electrical signals are *noisy*
  - You’ve seen this in the lab
  - Zoom in closely on a low-amplitude sinusoid with the scope (even one supplied directly from the function generator) – it won’t look like a perfectly clean sinusoid
Noise

- We will use the term *noise* to mean any electrical signal that interferes with or corrupts a signal we are trying to measure.

- Noise has many sources:
  - Measurement instruments themselves
  - 60Hz power line interference
  - Electrical components – resistors, transistors, etc.
  - Wireless LAN, fluorescent lights, computers, etc.

- We’d like to be able to remove, or filter out, this noise
  - Improve the accuracy of measurements
  - Often possible, if we know the *frequency characteristics* of the signal and the noise
Filtering Noise

- We’ll learn how to design filters to remove noise

First, we must introduce two important concepts:
- *Frequency-domain representation of electrical signals*
  - What is meant by “frequency characteristics” of an electrical signal?
- *Frequency response of linear systems*
  - How does a linear system (e.g. a filter) behave as a function of frequency?
Frequency Spectrum
Frequency Domain

- We are accustomed to looking at electrical signals in the **time domain**
  - Amplitude plotted as a *function of time*

- Can also be represented in the **frequency domain**
  - Amplitude plotted as a *function of frequency*
  - *Frequency spectrum*
    - Describes the *frequency content* of a signal
  - Can think of signals as a sum of different frequency sinusoids
    - What frequencies (sinusoids) are present
Frequency Spectrum

- *Frequency spectrum*
  - An *amplitude* vs. *frequency* plot
  - X-axis is frequency – not time
  - Y-axis is amplitude
  - Amplitude units may be in *decibels* (dB)

- Shows the relative amount of energy at each frequency

- Time-domain plot and frequency spectrum are alternate representations of the same signal
Frequency Spectra – Examples

- **Single sinusoid:** \( v(t) = 1V \cos(2\pi \cdot 800Hz \cdot t) \)

- **Sum of three sinusoids:**
  \[
  v(t) = 1V[\cos(2\pi \cdot 800Hz \cdot t) + \cos(2\pi \cdot 1200Hz \cdot t) + \cos(2\pi \cdot 2000Hz \cdot t)]
  \]
Frequency Spectra – Examples

- **White noise:**
  - Time-domain
  - Frequency-domain

- **Band-limited (colored) noise:**
  - Time-domain
  - Frequency-domain
Consider the following scenario:

- Measuring a sensor output in the lab
- Know the signal is roughly sinusoidal
- Suspected frequency: \( \sim 1 \text{ kHz} \)

Same signal in the frequency domain:

Three interfering tones

- All near 100 kHz

Can now design a filter to remove the noise:

Measured signal corrupted by noise/interference

- Difficult to identify the interfering signal from the time-domain plot
Fourier Transform

- **Fourier transform**
  - Transforms a time-domain representation to a frequency spectrum

  \[ V(\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt \]

- **Inverse Fourier transform**
  - Transforms from the frequency domain to the time domain

  \[ v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\omega) e^{j\omega t} d\omega \]

- A mathematical transform
  - Two different ways of looking at the same signal
  - A *change in perspective* not a change of the signal itself
Frequency Response of Linear Systems
Linear systems (electrical, mechanical, etc.) can be described by their \textit{frequency responses}

- Frequency response
  - Ratio of the system output phasor to the system input phasor
  - In general, a \textit{complex function of frequency}

\[
H(\omega) = \frac{Y}{X} = \frac{Y(\omega)}{X(\omega)}
\]

- Complex-valued – has both magnitude and phase
  - Magnitude: ratio of output to input magnitudes
    - \textit{Gain} of the system
  - Phase: difference in phase between output and input
    - \textit{Phase shift} through the system
Frequency Response – Bode Plots

- **Frequency response**
  - Description of system behavior as a function of frequency
  - **Gain** and **phase**

- Represented graphically – formatted as a **Bode plot**
  - Magnitude plot on top, phase plot below
  - Logarithmic frequency axes
  - Magnitude usually has units of decibels (dB)
  - Phase has units of degrees
Bode Plots

Units of magnitude are dB

Logarithmic frequency axes

Units of phase are degrees

Magnitude plot on top

Phase plot below
Decibels - dB

- Frequency response gain most often expressed and plotted with units of **decibels** (dB)
  - A *logarithmic* scale
  - Provides detail of very large and very small values on the same plot
  - Commonly used for *ratios* of powers or amplitudes

- Conversion from a linear scale to dB:
  \[ |H(\omega)|_{dB} = 20 \cdot \log_{10}(|H(\omega)|) \]

- Conversion from dB to a linear scale:
  \[ |H(\omega)| = 10^{\frac{|H(\omega)|_{dB}}{20}} \]
Decibels – dB

- Multiplying two gain values corresponds to adding their values in dB
  - E.g., the overall gain of cascaded systems
    \[ |H_1(\omega) \cdot H_2(\omega)|_{dB} = |H_1(\omega)|_{dB} + |H_2(\omega)|_{dB} \]

- Negative dB values corresponds to sub-unity gain
- Positive dB values are gains greater than one

<table>
<thead>
<tr>
<th>dB</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>1000</td>
</tr>
<tr>
<td>40</td>
<td>100</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>dB</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>-3</td>
<td>1/√2 = 0.707</td>
</tr>
<tr>
<td>-6</td>
<td>0.5</td>
</tr>
<tr>
<td>-20</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Value of Logarithmic Axes - dB

- Gain axis is linear in dB
  - A logarithmic scale
  - Allows for displaying detail at very large and very small levels on the same plot

- Gain plotted in dB
  - Two resonant peaks clearly visible

- Linear gain scale
  - Smaller peak has disappeared
Value of Logarithmic Axes - Frequency

- Frequency axis is logarithmic
  - Allows for displaying detail at very low and very high frequencies on the same plot

- Log frequency axis
  - Can resolve frequency of both resonant peaks

- Linear frequency axis
  - Lower resonant frequency is unclear
Interpreting Bode Plots

Bode plots tell you the gain and phase shift at all frequencies: choose a frequency, read gain and phase values from the plot.

For a 10KHz sinusoidal input, the gain is 0dB (1) and the phase shift is 0°.

For a 10MHz sinusoidal input, the gain is -32dB (0.025), and the phase shift is -176°.
Interpreting Bode Plots

Bode Plot

Magnitude [dB]

Phase [deg]

Frequency [Hz]

Sinusoidal Response

x(t), y(t)

Time [sec]

0 20 40 60 80 100
Example Problems
A measured signal has the frequency spectrum shown here. Assuming the larger signal component has an amplitude of 500 mV, and that both signal components are in phase, write a time-domain expression for the measured signal.
Determine the frequency response function, $H(\omega)$, for the following circuit.

What are the circuit’s gain and phase at 200 kHz?
The input to a circuit with the following Bode plot is
\[ v_i(t) = 1.2V \cdot \cos(2\pi \cdot 10kHz \cdot t) \]
What is the output, \( v_o(t) \)?
Filters are classified by the ranges of frequencies they pass and those they filter out.
Filter Operation

- **Frequency spectrum** describes frequency content of electrical signals.
- **Frequency response** describes system (circuit) gain and phase at different frequencies.

Can design circuits (i.e. filters) to have high gain at desirable frequencies and low gain at undesirable frequencies.

- Want to filter out high frequencies?
  - Design a filter with low gain at high frequencies and high gain at low frequencies.

- Want to filter out all signals between 1 MHz and 10 MHz?
  - Design a filter with low gain in this range and high gain everywhere else.
Types of Filters

- Filters are classified according to the ranges of frequencies they pass and those they filter out.

  - **Low pass filters:** pass low frequencies, filter out high frequencies
  
  - **High pass filters:** pass high frequencies, filter out low frequencies
  
  - **Band pass filters:** pass only a range of frequencies, filter out everything else
  
  - **Band stop (notch) filters:** filter out only a certain range of frequencies, pass all others
Ideal Filters

- **Ideal filter** gain characteristics:
  - **Unity gain** in the **pass band**
    - Range of frequencies to be passed
  - **Zero gain** in the **stop band**
    - Range of frequencies to be filtered out
  - **Abrupt transition** between pass band and stop band

- Signals with frequency components in the pass band pass through the filter unaltered
- Signals with frequency components in the stop band are completely filtered out – removed from the signal
Ideal Filters – Magnitude Response

- Ideal Low Pass Filter
- Ideal High Pass Filter
- Ideal Band Pass Filter
- Ideal Band Stop Filter

- Note the use of a linear gain scale here
  - Stop band gain of zero translates to $-\infty$ dB

- Ideal filters often referred to as **brick wall filters**
**Real Filters – Magnitude Response**

**Magnitude response for a real low pass filter:**

Pass band edge is freq. at which gain is down by 3 dB – the **-3 dB frequency**. This is the filter’s **bandwidth**.

*Roll-off rate* between pass band and stop band depends on the type of filter – particularly, the **order** of the filter.
First-Order Passive Filters

*First-order* – only one energy-storage element

*Passive* – contain only resistors and capacitors or inductors – no opamps or transistors
Filters as Voltage Dividers

- Already familiar with resistive voltage dividers:
  \[ v_o = v_s \frac{R_2}{R_1 + R_2} \]

- Frequency response function:
  \[ H(\omega) = \frac{V_o}{V_s} = \frac{R_2}{R_1 + R_2} \]

- A real constant – independent of frequency
  - Same gain at all frequencies
  - No phase shift at any frequency

- Now consider a circuit whose resistances have been replaced with impedances:
  \[ H(\omega) = \frac{V_o}{V_s} = \frac{Z_2}{Z_1 + Z_2} \]

- Frequency response is now a complex function of frequency
  - Gain and phase vary as a function of frequency
  - Basis for the design of first-order filters
RC Low Pass Filter
Now, let $Z_1$ be resistive and $Z_2$ be capacitive

Frequency response:

$$H(\omega) = \frac{V_o}{V_s} = \frac{Z_2}{Z_1 + Z_2} = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$H(\omega) = \frac{1}{1 + j\omega RC}$$

Recall from ENGR 201 that the transient response of this same circuit is characterized by its time constant, $\tau = RC$

In the frequency domain, this is the *corner frequency* or *break frequency*

$$\omega_c = \frac{1}{\tau} = \frac{1}{RC}$$

and

$$f_c = \frac{1}{2\pi RC}$$

The frequency at which gain is down by 3 dB

The -3 dB frequency

Frequency at which the magnitude of R and C impedances are equal
To gain insight into the behavior of this filter circuit, consider two limiting cases:

- **As } f \rightarrow 0,**
  - Capacitor $\rightarrow$ open circuit
  - $i(t) \rightarrow 0$
  - $v_o \rightarrow v_s$
  - **Gain $\rightarrow$ unity**

- **As } f \rightarrow \infty**
  - Capacitor $\rightarrow$ short circuit
  - $v_o$ shorted to ground
  - **Gain $\rightarrow$ zero**
RC Low Pass Filter – Bode Plot
Gain is -3 dB at $f_c$, the **bandwidth** of the filter.

Low-frequency asymptote at 0dB.

Response rolls off at -20dB/decade, or -6dB/octave above $f_c$.

*RC Low Pass Filter – Magnitude Response*
RC Low Pass Filter – Phase Response

- Phase approximation rolls off at -45°/decade around $f_c$
- Phase is -45° at $f_c$
- One decade below $f_c$
- High-frequency asymptote at -90°
- One decade above $f_c$
- Low-frequency asymptote at 0°
Known slope can be used to relate gains at different frequencies

\[
\text{Slope} \left[ \frac{\text{dB}}{\text{dec}} \right] = \frac{|H(f_2)|_{dB} - |H(f_1)|_{dB}}{\log_{10}(f_2) - \log_{10}(f_1)}
\]

For example:

\[
-20 \frac{\text{dB}}{\text{dec}} = \frac{|H(7 MHz)| - |H(1 MHz)|_{dB}}{\log_{10}(7 MHz) - \log_{10}(1 MHz)}
\]

\[
-20 \frac{\text{dB}}{\text{dec}} = \frac{|H(7 MHz)| - (-14 dB)}{6.845 - 6}
\]

\[
|H(7 MHz)| = -30.9 dB
\]
Simple first-order RC low pass filters provide a quick and easy way to remove noise from electrical signals.

Consider for example a *dual-tone multi-frequency (DTMF)* signal:
- Touch-tone telephone signal (key 5 in this example)
- Tone is the sum of two sinusoids (key 5 = 1336Hz and 770Hz)
- Pressing the “5” key generates the DTMF signal
- Noise on the DTMF signal makes decoding impossible
- Filter noise to enabling decoding
Key number 5 is pressed
  - DTMF signal generated
    - Sum of 770 Hz and 1336 Hz sinusoids

Decoder at the receiving end decodes the DTMF signal and determines that a 5 was pressed
Consider a more realistic scenario
- DTMF signal corrupted by a significant amount of noise

The decoder is no longer able to determine that a 5 was pressed
The goal is to filter the received signal so that the decoder can accurately interpret the DTMF signal.

Designing the low pass filter:
- White noise:
  - Flat frequency spectrum
  - Equal power at all frequencies
- DTMF frequency range: 697 Hz – 1633 Hz
- Want to attenuate as much noise as possible
- Want to attenuate DTMF signals as little as possible
- RC LPF with corner frequency at 10 kHz will limit DTMF-band attenuation to < 0.2 dB
RC LP Filter – Application Example

- **RC LPF design**
  - Need to select R and C to set the corner frequency

  \[
  f_c = \frac{1}{2\pi RC} = 10 \text{ kHz}
  \]

  - Say we have a 0.1 \( \mu \text{F} \) capacitor available
  - Solve for R

  \[
  R = \frac{1}{2\pi f_c C}
  \]

  \[
  R = \frac{1}{2\pi \cdot 10 \text{ kHz} \cdot 0.1 \mu \text{F}} = 159 \Omega
  \]

  \[
  R = 159 \Omega, \quad C = 0.1 \mu \text{F}
  \]
RC LP Filter – Application Example

Bode plot of the resulting filter:

DTMF signals lie in this range – passed through the filter with little attenuation.
RC LP Filter – Application Example

- Filter allows DTMF signal to pass mostly unaltered
- Noise below 10 kHz is mostly passed through
- Noise above 10 kHz is attenuated, but not removed
- Received signal is not noiseless, but clean enough to be decoded
Example Problems
Design a filter to pass the desired 500 Hz signal and to attenuate the unwanted 100 kHz by 40 dB.

What is the signal-to-noise ratio (SNR) at the output of the filter?
RC High Pass Filter
Now, swap the locations of the resistor and capacitor

- **Frequency response:**
  
  $$H(\omega) = \frac{V_o}{V_s} = \frac{Z_2}{Z_1 + Z_2} = \frac{R}{R + 1/j\omega C}$$

  $$H(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

- **Corner frequency is the same as for the low pass filter**
  
  $$\omega_c = \frac{1}{\tau} = \frac{1}{RC}$$

  and

  $$f_c = \frac{1}{2\pi RC}$$

- **The frequency at which gain is down by 3 dB**
- **Frequency at which the capacitor impedance magnitude is equal to the resistor impedance magnitude**
- **Now, gain is constant above** $f_c$ **and rolls off below** $f_c$
RC High Pass Filter

To gain insight into the behavior of this filter circuit, consider two limiting cases:

- As $f \rightarrow 0$,
  - Capacitor $\rightarrow$ open circuit
  - $i(t) \rightarrow 0$
  - $v_o \rightarrow 0$
  - Gain $\rightarrow$ zero

- As $f \rightarrow \infty$
  - Capacitor $\rightarrow$ short circuit
  - $v_o$ shorted to $v_s$
  - Gain $\rightarrow$ unity
RC High Pass Filter – Bode Plot
RC High Pass Filter – Magnitude Response

Magnitude response rolls off at 20 dB/decade, or 6 dB/octave below $f_c$. Gain is -3dB at $f_c$. High-frequency asymptote at 0 dB.
RC High Pass Filter – Phase Response

\[ f_c = \frac{1}{kRHC} \]

Phase approximation rolls off at -45°/decade around \( f_c \).

Phase is +45° at \( f_c \).

Low-frequency asymptote at +90°.

High-frequency asymptote at 0°.

One decade below \( f_c \).

One decade above \( f_c \).
High pass filters are useful for removing low-frequency content, including DC, from electrical signals.

For example, consider the following scenario:

- Instrumented a flow loop in the lab
  - Pumps, temperature sensors, pressure sensors, and flow meters
- Flow meter output seems to be erroneous every ~1 msec
- Suspected cause: coupled through the +12V power supply from one of the pumps
- Want to measure the flow meter’s +12V power supply with a channel on our data acquisition system (DAQ)
  - Dynamic range of DAQ input: ±5 V
- Use a high-pass filter to remove the +12V DC component from the power supply voltage
RC HP Filter – Application Example

- Want to a +12 V supply with a ±5 V DAQ input
- High pass filter will remove the DC component of the supply voltage

- High pass filter used to remove DC signal components
- Couples only AC signal components to the DAQ input
  - AC coupling
  - Similar to the AC coupling setting on the scopes in the lab
RC HP Filter – Application Example

- High pass filter design
  - Want to remove DC
    - Low corner frequency
  - High RC time constant
    - Large R and C
  - Arbitrarily set $f_c = 10 \text{ Hz}$

- DAQ system
  - Datasheet says $R_{in} = 10 \text{ M}\Omega$
    - Let $R_{in}$ be the filter resistance

- Calculate C to get desired $f_c$

\[
C = \frac{1}{2\pi f_c R} = \frac{1}{2\pi \cdot 10\text{ Hz} \cdot 10\text{ M}\Omega}
\]

\[
C = 15.9 \text{ nF}
\]

- Or anything in that neighborhood
- Not critical – just want to block DC
**RC HP Filter – Application Example**

**RC high pass filter:**

![RC high pass filter circuit diagram](image)

**High pass filter Bode plot:**

The +12V DC component of the power supply voltage is completely removed.

![High pass filter Bode plot graph](image)
The noisy +12V power supply at the malfunctioning flow meter:

- DC value of signal is +12 V
- Outside ±5 V DAQ input dynamic range

High pass filter output – AC coupled power supply voltage:

- DC value of signal is now 0 V
- Within ±5 V DAQ input range
- Glitches clearly measured with the DAQ
Oscilloscopes – AC Coupling

- Scope inputs allow you to select between DC and AC coupling
  - Usually under the *channel* menu
  - **DC coupling**: input signal is terminated in $1\,\text{M}\Omega$ and connected directly to the preamp and ADC in the scope
  - **AC coupling**: input signal is switched through a capacitor that forms a high pass filter with the $1\,\text{M}\Omega$ input resistor
    - $f_c \approx 3.5 \, \text{Hz}$ – removes DC
    - Useful for looking at power supply ripple, etc.
Oscilloscopes – AC Coupling

**High-impedance scope front-end:**

**Configured for DC coupling:**

**Configured for AC coupling:**
RL Filters
First-order RL filters

- Can also use *inductors* to make RL low pass and high pass filters
- Capacitors are usually preferable for simple first-order filters
  - Smaller
  - Cheaper
  - Draw no DC current

**RL low pass filter:**

\[ f_c = \frac{R}{2\pi L} \]

**RL high pass filter:**

\[ f_c = \frac{R}{2\pi L} \]
RL Low Pass Filter

- **RL low pass filter**

  ![RL low pass filter circuit diagram]

  - **Frequency response:**
    
    \[ H(\omega) = \frac{V_o}{V_s} = \frac{Z_2}{Z_1 + Z_2} = \frac{R}{R + j\omega L} \]

  - **Corner frequency is one over the time constant**
    
    \[ \omega_c = \frac{1}{\tau} = \frac{R}{L} \]
    
    and
    
    \[ f_c = \frac{R}{2\pi L} \]

  - **The frequency at which gain is down by 3 dB**

  - **Frequency at which the inductor impedance magnitude is equal to the resistor impedance magnitude**

  - **Bode plot identical to that of the RC low pass filter**
    
    As it is for all first-order low pass systems
Again consider the filter’s behavior for two limiting cases:

- **As** \( f \to 0 \),
  - Inductor \( \to \) short circuit
  - \( v_o \) shorted to \( v_s \)
  - \textit{Gain} \( \to \text{unity} \)

- **As** \( f \to \infty \)
  - Inductor \( \to \) open circuit
  - \( i(t) \to 0 \)
  - \( v_o \to 0 \)
  - \textit{Gain} \( \to \text{zero} \)
Now, swap the locations of the resistor and inductor

Frequency response:

\[ H(\omega) = \frac{V_o}{V_s} = \frac{Z_2}{Z_1 + Z_2} = \frac{j\omega L}{R + j\omega L} \]

Corner frequency is the same as for the low pass filter

\[ \omega_c = \frac{1}{\tau} = \frac{R}{L} \quad \text{and} \quad f_c = \frac{R}{2\pi L} \]

Bode plot is identical to that of the RC high pass filter

Gain is constant \textit{above} \( f_c \) and rolls off \textit{below} \( f_c \)
Again, consider the two limiting frequency cases

- As $f \to 0$,
  - Inductor $\to$ short circuit
  - $v_o$ shorted to ground
  - *Gain* $\to$ zero

- As $f \to \infty$
  - Inductor $\to$ open circuit
  - $i(t) \to 0$
  - $v_o \to v_s$
  - *Gain* $\to$ unity
Audio Filter Demo
Analog Discovery Instrument

- 2-chan. Scope
  - 14-bit, 100MSa/s
  - 5MHz bandwidth
- 2-chan. function generator
  - 14-bit, 100MSa/s
  - 5MHz bandwidth
- 2-chan. spectrum analyzer
- Network analyzer
- Voltmeter
- ±5V power supplies
- 16-chan. logic analyzer
- 16-chan. digital pattern generator
- USB connectivity
Analog Discovery – Audio Demo

- Demo board plugs in to Analog Discovery module
- Summation of multiple tones
- Optional filtering of audio signal
- 3.5 mm audio output jack
Analog Discovery – Audio Demo