SECTION 3: SECOND-ORDER FILTERS

ENGR 202 – Electrical Fundamentals II

Introduction

Second-Order Circuits

- ***Order*** of a circuit (or system of any kind)
  - ***Number of independent energy-storage elements***
  - ***Order of the differential equation*** describing the system

- ***Second-order circuits***
  - Two energy-storage elements
  - Described by second-order differential equations

- We will primarily be concerned with ***second-order RLC circuits***
  - Circuits with a resistor, an inductor, and a capacitor

Second-Order Circuits

- In this and the following section of notes, we will look at second-order RLC circuits from two distinct perspectives:
  - <u>Section 3</u>
    - Second-order ***filters***
    - ***Frequency-domain*** behavior
  - <u>Section 4</u>
    - Second-order ***transient response***
    - ***Time-domain*** behavior

5 Second-Order Filters

Second-Order Filters

- ***First-order filters***
  - Roll-off rate: ***20 dB/decade***
- This roll-off rate determines ***selectivity***
  - Spacing of pass band and stop band
  - Spacing of *passed* frequencies and *stopped* or *filtered* frequencies
- ***Second-order filters***
  - Roll-off rate: ***40 dB/decade***
- In general:
  - Roll-off = $N \cdot 20\ dB/dec$, where N is the filter order

Resonance

- ***Resonance***
  - Tendency of a system to oscillate at certain frequencies – ***resonant frequencies*** – often with larger amplitude than any input
  - Phenomenon that occurs in all types of dynamic systems (mechanical, electrical, fluid, etc.)

- Examples of resonant mechanical systems:
  - Mass on a spring
  - Pendulum, playground swing
  - Tacoma Narrows Bridge

Electrical Resonance

- ***Electrical resonance***
  - Cancellation of ***reactances*** (or ***susceptances***), resulting in purely resistive network impedance
  - Occurs at ***resonant frequencies***
  - Second- and higher-order circuits

- ***Reactances*** (***susceptances***) ***cancel*** – sum to zero ohms (siemens)
  - Inductive reactance is positive – (susceptance is negative)
  - Capacitive reactance is negative – (susceptance is positive)

- Voltages/currents in the circuit may be much larger than source voltages/currents

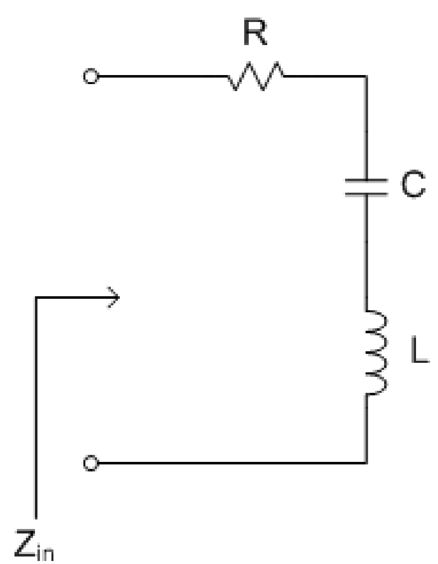
- We'll take a look at resonance in two classes of circuits:
  - ***Series*** resonant circuits
  - ***Parallel*** resonant circuits

Series Resonant Circuits

Series Resonant RLC Circuit

- Series RLC circuit
  - Second-order – one capacitor, one inductor
  - Circuit will exhibit resonance

Impedance of the network:



$$Z_{in}(\omega) = R + \frac{1}{j\omega C} + j\omega L = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

At the resonant frequency, ω_0 or f_0:

$$X_L + X_C = 0 \;\rightarrow\; X_L = -X_C$$

$$\omega_0 L = \frac{1}{\omega_0 C} \;\rightarrow\; \omega_0^2 = \frac{1}{LC}$$

so

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}} \quad \text{and} \quad \boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

and

$$\boxed{Z_{in}(\omega_0) = R}$$

Series RLC Circuit – Quality Factor

- ***Quality factor***, Q_s
  - Ratio of inductive reactance *at the resonant frequency* to resistance

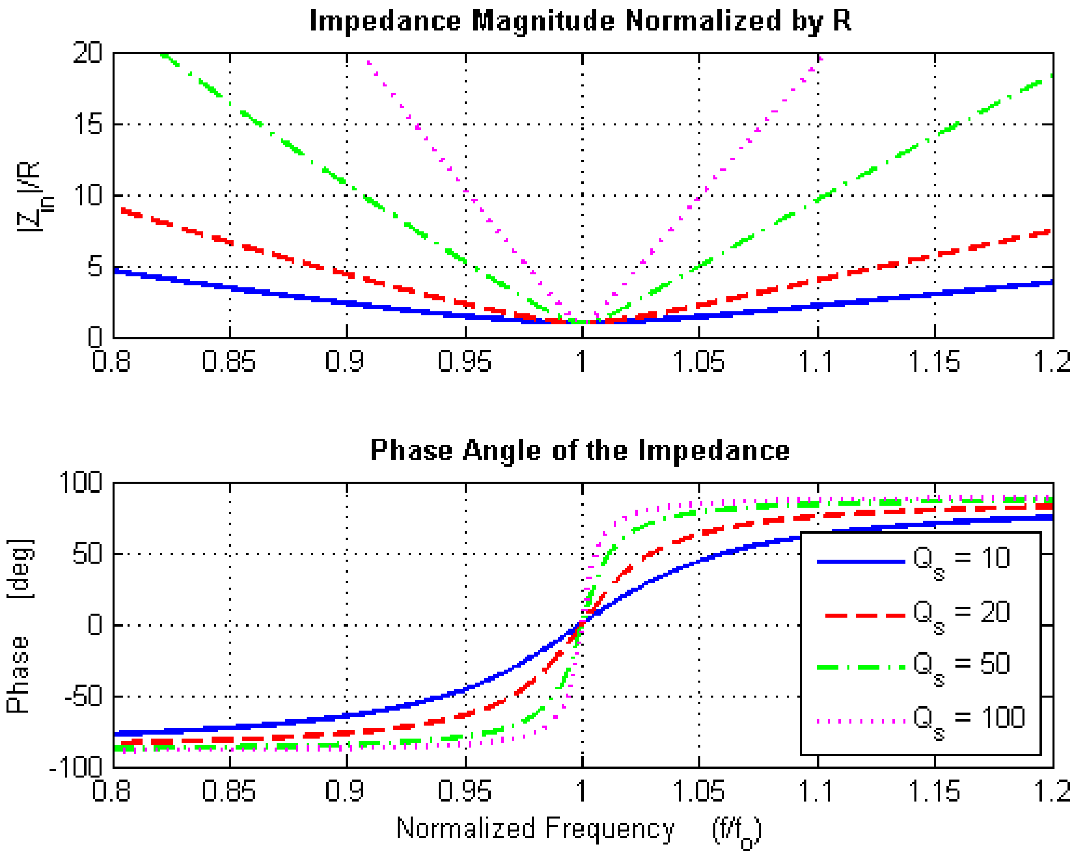
$$\boxed{Q_s = \frac{\omega_0 L}{R} = \frac{2\pi f_0 L}{R}}$$

  - At resonance, inductive and capacitive reactances (magnitudes) are equal, so

$$\boxed{Q_s = \frac{1}{\omega_0 RC} = \frac{1}{2\pi f_0 RC}}$$

  - The ratio of voltage magnitude across the inductor or capacitor to the voltage across the whole RLC network *at resonance*
  - A measure of the *sharpness* of the resonance

Series RLC Circuit – Z_{in} vs. Q_s



- At $f = f_0$
  - $|Z_{in}| = R$
  - $\angle Z_{in} = 0°$
- Q determines *sharpness* of the resonance
  - Higher Q yields faster transition from capacitive, through resistive, to inductive regions
- To increase Q:
  - Increase L
  - Reduce R and/or C

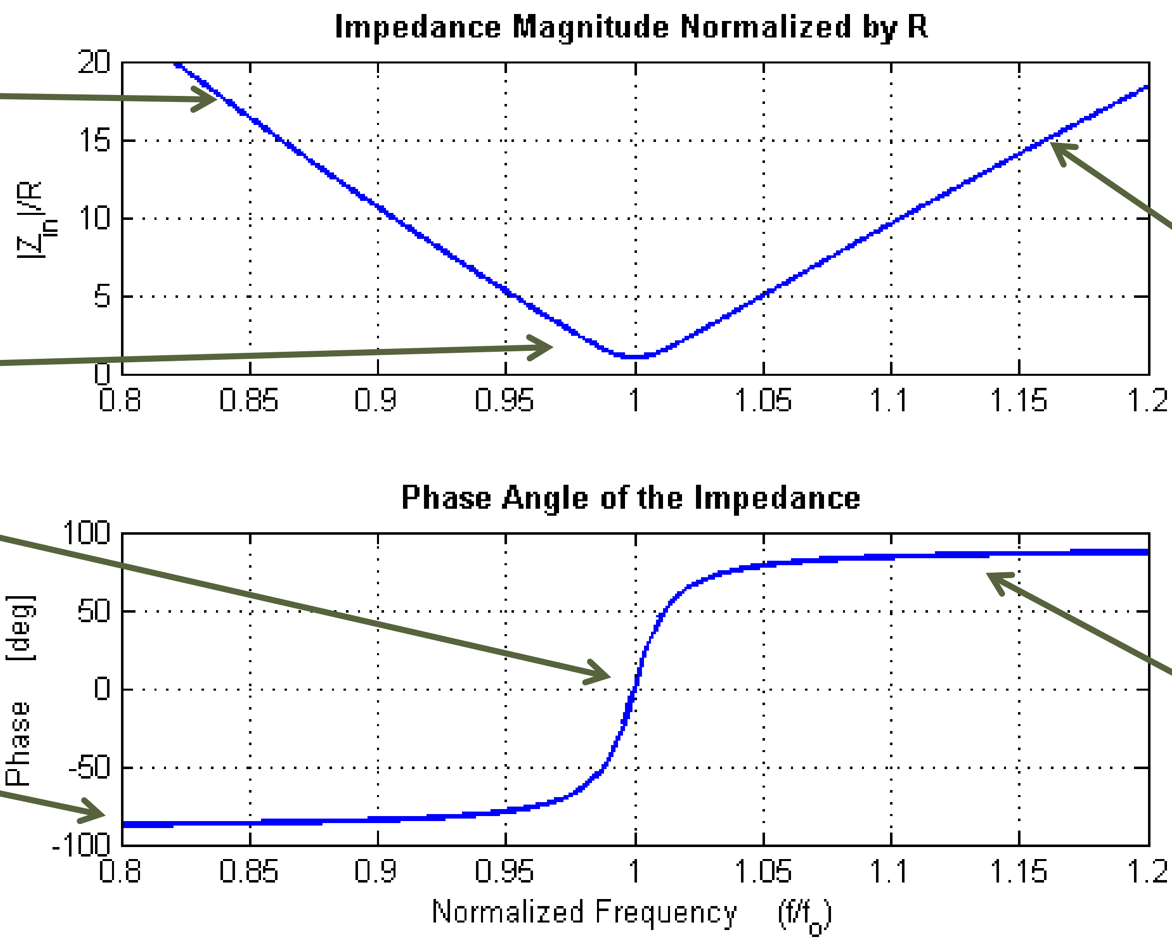
Series RLC Circuit – Z_{in}

Understanding the impedance of a series resonant circuit

Capacitor impedance goes up as f goes down.

$f = f_0$:
$Z_{in} = R$
Z_{in} is real
$\angle Z_{in} = 0°$

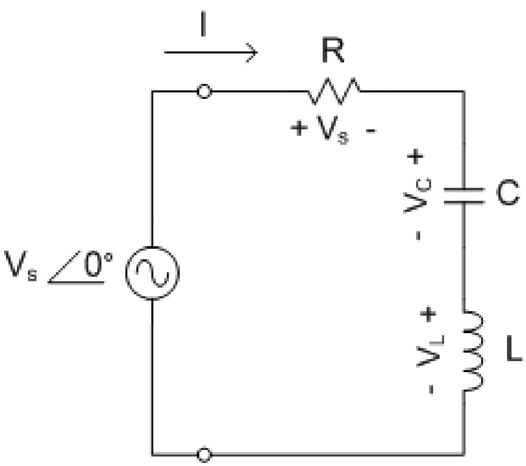
$f \ll f_0$:
$\angle Z_{in} = -90°$
Z_{in} looks capacitive



Inductor impedance goes up as f goes up.

$f \gg f_0$:
$\angle Z_{in} = +90°$
Z_{in} looks inductive

Series RLC Circuit – Voltages and Currents



- At $\omega_0, Z_{in} = R$, so the current phasor is

$$\mathbf{I} = \frac{\mathbf{V_S}}{R} = \frac{V_s}{R}\angle 0°$$

- ***Capacitor voltage*** at resonance:

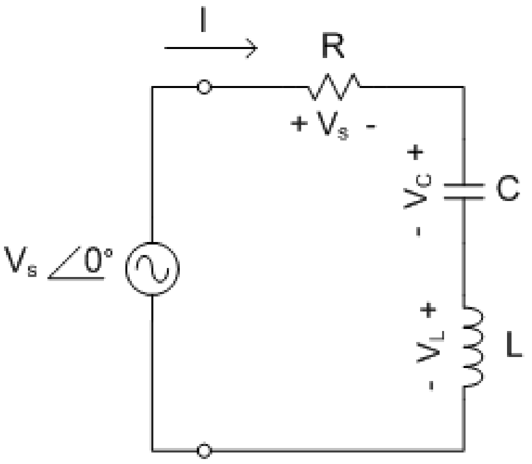
$$\mathbf{V_C} = \frac{\mathbf{I}}{j\omega_0 C} = \frac{V_s\angle 0°}{\omega_0 RC\angle 90°} = \frac{V_s}{\omega_0 RC}\angle -90°$$

- Recalling the expression for quality factor of a series resonant circuit, we have

$$\mathbf{V_C} = Q_s \cdot V_s\angle -90°$$

  - The voltage across the capacitor is the source voltage multiplied by the quality factor and phase shifted by $-90°$

Series RLC Circuit – Voltages and Currents



- The ***inductor voltage*** at resonance:

$$\mathbf{V_L} = \mathbf{I} \cdot j\omega_0 L = \frac{V_s\angle 0° \cdot \omega_0 L\angle 90°}{R}$$

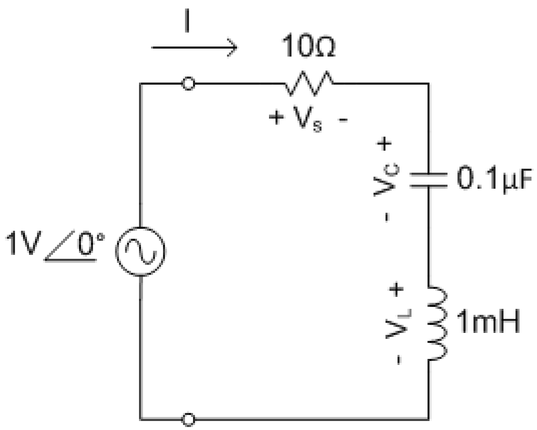
$$\mathbf{V_L} = \frac{V_s \cdot \omega_0 L\angle 90°}{R}$$

- Again, substituting in the expression for quality factor gives

$$\mathbf{V_L} = Q_s \cdot V_s\angle + 90°$$

  - The voltage across the inductor is the source voltage multiplied by the quality factor and phase shifted by $+90°$
- Capacitor and inductor voltage at resonance:
  - Equal magnitude
  - $180°$ out of phase – opposite sign – they cancel

Series RLC Circuit – Voltages and Currents



- Now assign component values
  - The resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = 100\,\frac{krad}{sec}$$

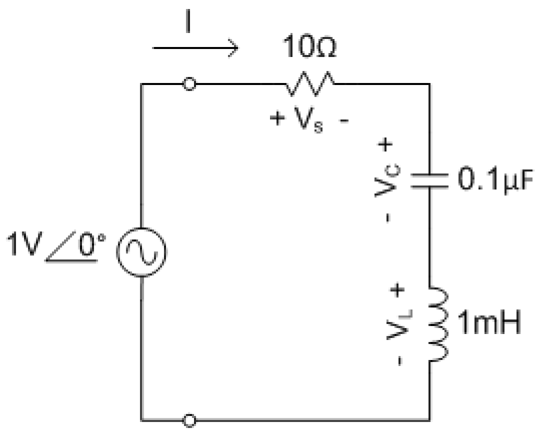
  - The quality factor is

$$Q_s = \frac{\omega_0 L}{R} = \frac{100\,\frac{krad}{sec} \cdot 1\;mH}{10\;\Omega} = 10$$

- The current phasor at the resonant frequency is

$$\mathbf{I} = \frac{\mathbf{V_s}}{R} = \frac{1\;V\angle 0°}{10\;\Omega} = 100\angle 0°\;mA$$

Series RLC Circuit – Voltages and Currents



- The capacitor voltage at the resonant frequency is

$$\mathbf{V_C} = \frac{\mathbf{I}}{j\omega_0 C} = \frac{V_s}{\omega_0 RC}\angle -90° = Q_s \cdot V_s\angle -90°$$

$$\mathbf{V_C} = 10 \cdot 1\,V\angle -90°$$

$$\mathbf{V_C} = 10\,V\angle -90°$$

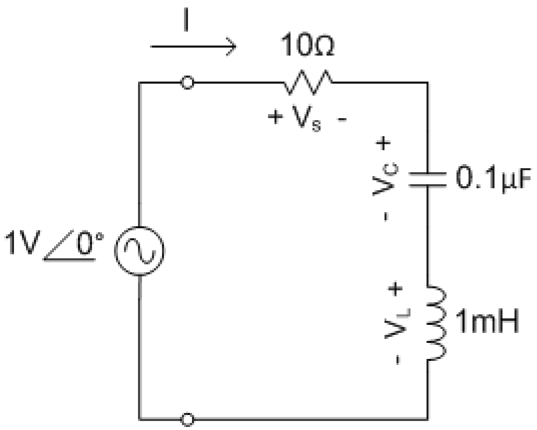
- The inductor voltage at the resonant frequency:

$$\mathbf{V_L} = \mathbf{I} \cdot j\omega_0 L = \frac{V_s \cdot \omega_0 L\angle 90°}{R} = Q_s \cdot V_s\angle +90°$$

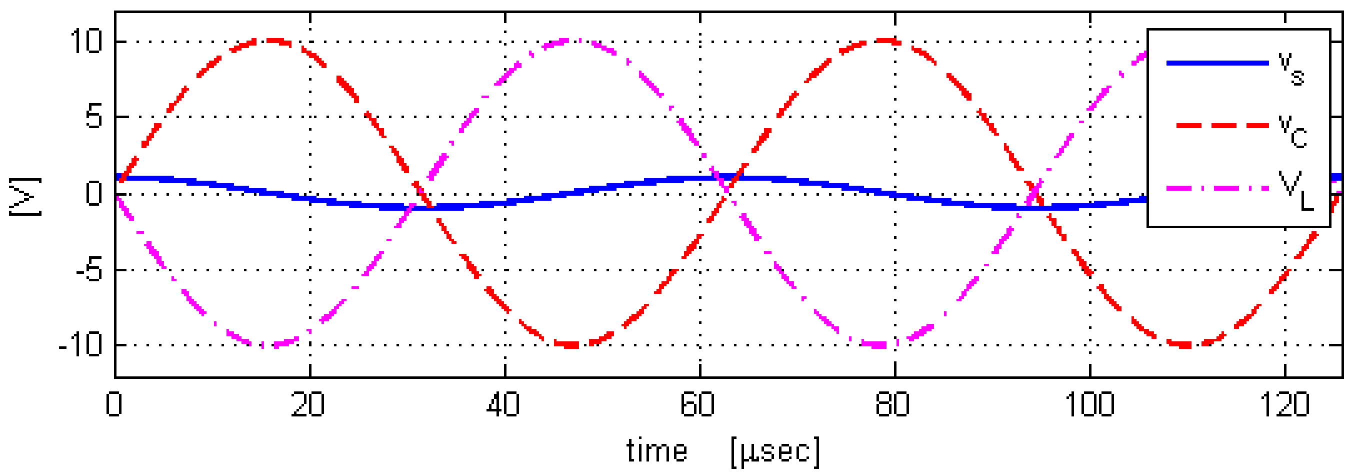
$$\mathbf{V_L} = 10 \cdot 1\,V\angle +90°$$

$$\mathbf{V_L} = 10\,V\angle +90°$$

Series RLC Circuit – Voltages and Currents



- $|\mathbf{V_S}| = 1\ V$
- $\angle\mathbf{I} = 0°$
- $|\mathbf{V_C}| = |\mathbf{V_L}| = Q_s|\mathbf{V_S}| = 10\ V$
- $|\mathbf{V_C}|$ and $|\mathbf{V_L}|$ are $180°$ out of phase
  - They cancel
  - KVL is satisfied



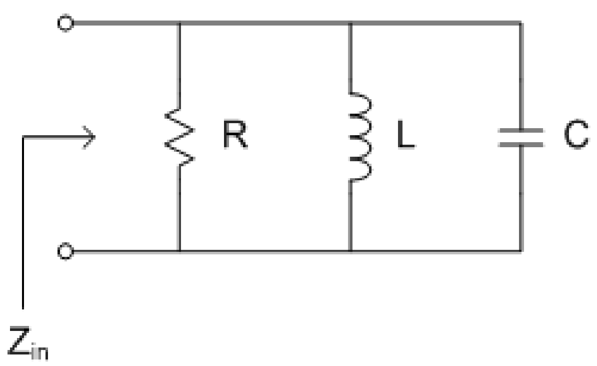
Parallel Resonant Circuits

Parallel Resonant RLC Circuit

- Parallel RLC circuit
  - Second-order – one capacitor, one inductor
  - Circuit will exhibit resonance

Impedance of the network:

$$Z_{in}(\omega) = \left[\frac{1}{R} + j\omega C + \frac{1}{j\omega L}\right]^{-1} = \left[\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)\right]^{-1}$$



At the resonant frequency, ω_0 or f_0:

$$B_C + B_L = 0 \rightarrow B_C = -B_L$$

$$\omega_0 C = \frac{1}{\omega_0 L} \rightarrow \omega_0^2 = \frac{1}{LC}$$

so

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}} \quad \text{and} \quad \boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

and

$$\boxed{Z_{in}(\omega_0) = R}$$

Parallel RLC Circuit – Quality Factor

- ***Quality factor***, Q_p
  - Ratio of inductive susceptance *at the resonant frequency* to conductance

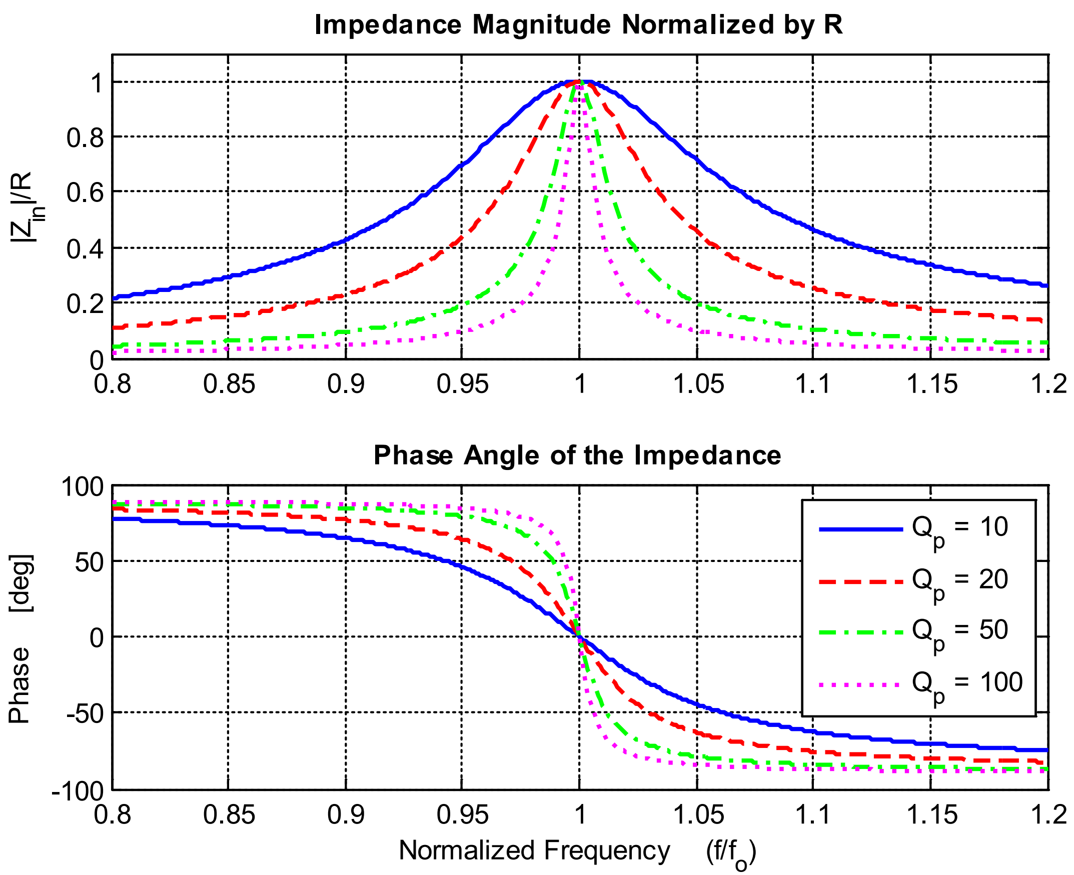
$$\boxed{Q_p = \frac{1/\omega_0 L}{1/R} = \frac{R}{\omega_0 L} = \frac{R}{2\pi f_0 L}}$$

  - At resonance, inductive and capacitive susceptances (magnitudes) are equal, so

$$\boxed{Q_p = \omega_0 RC = 2\pi f_0 RC}$$

  - The ratio of current magnitude through the inductor or capacitor to the current through the whole RLC network *at resonance*
  - A measure of the *sharpness* of the resonance

Parallel RLC Circuit – Z_{in} vs. Q_p



- At $f = f_0$
  - $|Z_{in}| = R$
  - $\angle Z_{in} = 0°$
- Q determines *sharpness* of the resonance
  - Higher Q yields faster transition from inductive, through resistive, to capacitive regions
- To increase Q:
  - Reduce L
  - Increase R and/or C

Parallel RLC Circuit – Z_{in}

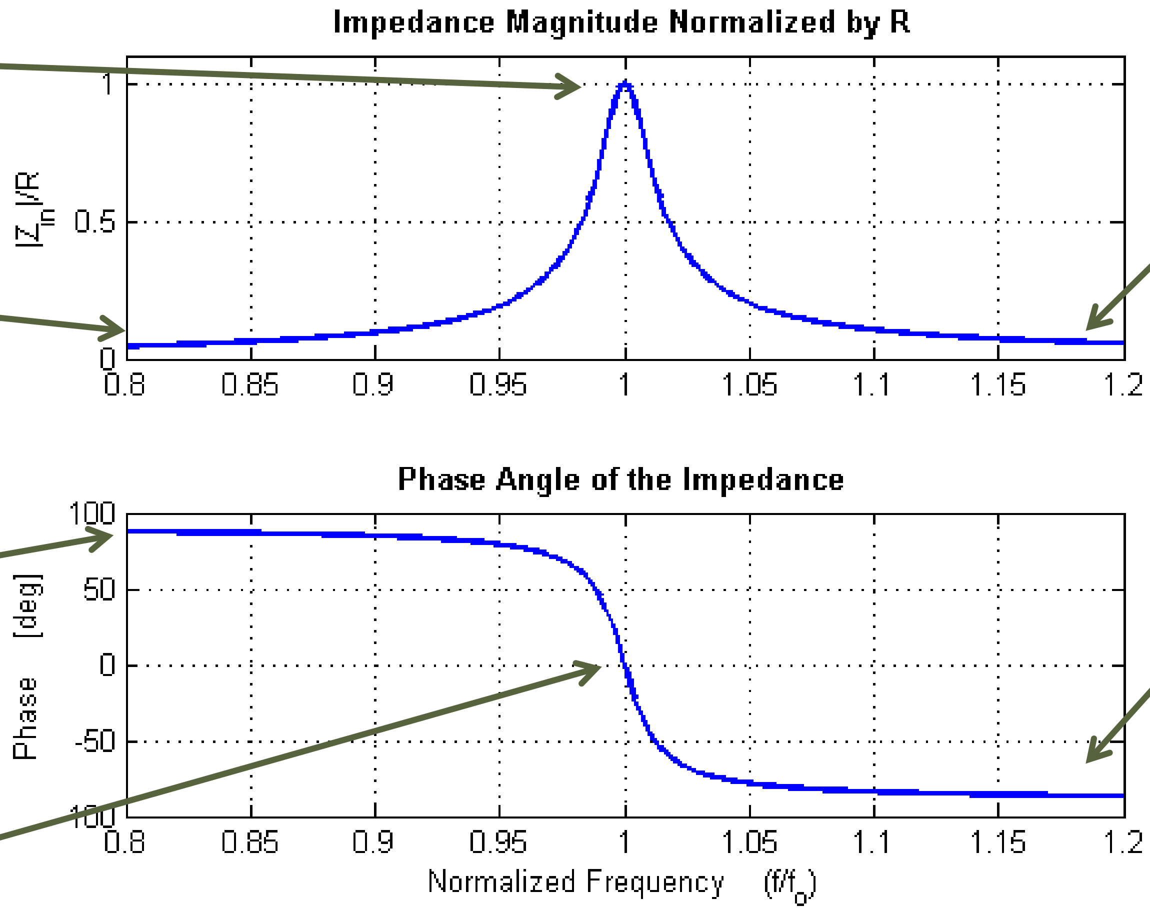
Understanding the impedance of a parallel resonant circuit

$f = f_0$:
$Z_{in} = R$
Z_{in} is real

Inductor tends toward a short as $f \rightarrow 0$

$f \ll f_0$:
$\angle Z_{in} = +90°$
Z_{in} looks inductive

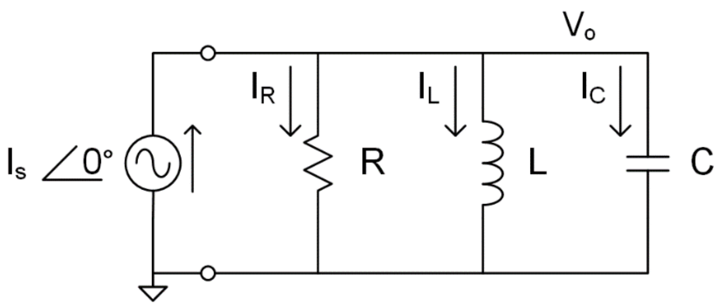
$f = f_0$:
Z_{in} is real
$\angle Z_{in} = 0°$



Capacitor tends toward a short as $f \rightarrow \infty$

$f \gg f_0$:
$\angle Z_{in} = -90°$
Z_{in} looks capacitive

Parallel RLC Circuit – Voltages and Currents



- Sinusoidal current source, $\mathbf{I_s}$
- At resonance, $Z_{in} = R$, so the voltage across the network is:

$$\mathbf{V_o} = \mathbf{I_s}R = I_s\angle 0° \cdot R$$

- Current through the ***capacitor*** at resonance:

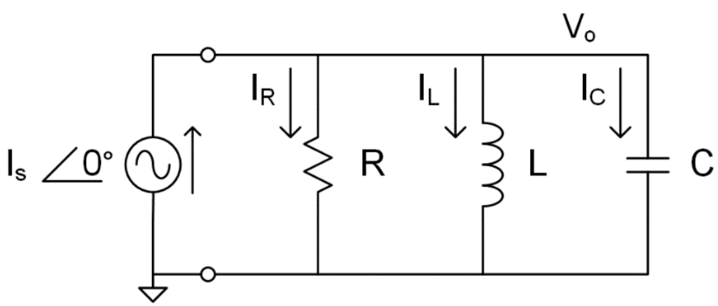
$$\mathbf{I_C} = \mathbf{V_o} \cdot j\omega_0 C = I_s R \cdot \omega_0 C\angle 90°$$

- Recalling the expression for quality factor of the parallel resonant circuit, we have

$$\mathbf{I_C} = Q_p \cdot I_s\angle 90°$$

  - The current through the capacitor is the source current multiplied by the quality factor and phase shifted by 90°

Parallel RLC Circuit – Voltages and Currents



- The ***inductor current*** at resonance:

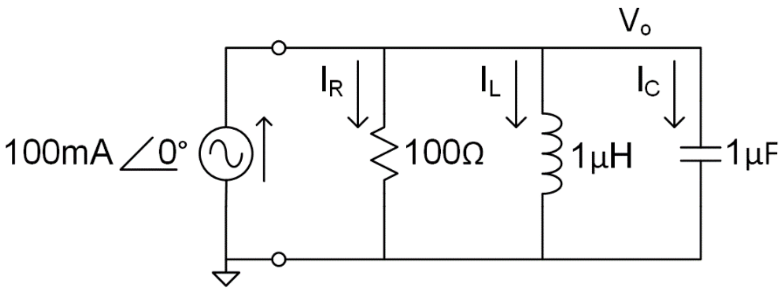
$$\mathbf{I_L} = \frac{\mathbf{V_o}}{j\omega_0 L} = \frac{I_s R}{\omega_0 L}\angle -90°$$

- Again, substituting in the expression for quality factor gives

$$\mathbf{I_L} = Q_p \cdot I_s \angle -90°$$

  - The current through the inductor is the source current multiplied by the quality factor and phase shifted by $-90°$
- Capacitor and inductor current at resonance:
  - Equal magnitude
  - $180°$ out of phase – opposite sign – they cancel

Parallel RLC Circuit – Voltages and Currents



- Now, assign component values
  - The resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = 1\,\frac{Mrad}{sec}$$

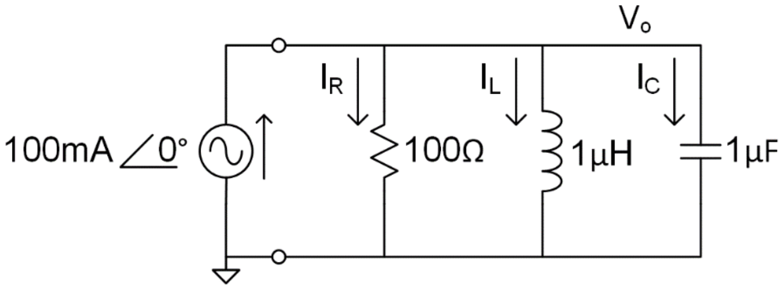
  - The quality factor is

$$Q_p = \frac{R}{\omega_0 L} = \frac{100\ \Omega}{1\,\frac{Mrad}{sec} \cdot 1\ \mu H} = 100$$

- The phasor for the voltage across the network at the resonant frequency is

$$\mathbf{V_o} = \mathbf{I_s} R = 100\ mA\angle 0° \cdot 100\ \Omega = 10\angle 0°\ V$$

Parallel RLC Circuit – Voltages and Currents



- The capacitor current at the resonant frequency is

$$\mathbf{I_C} = \mathbf{V_o} \cdot j\omega_0 C = I_s \cdot \omega_0 RC\angle 90°$$

$$\mathbf{I_C} = Q_p \cdot I_s\angle 90° = 100 \cdot 100\ mA\angle 90°$$

$$\mathbf{I_C} = 10\ A\angle 90°$$

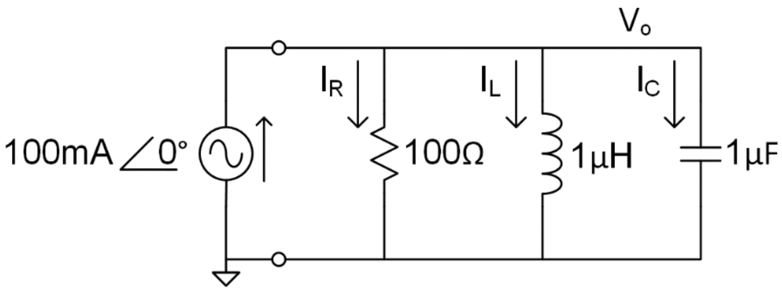
- The inductor current at the resonant frequency:

$$\mathbf{I_L} = \frac{\mathbf{V_o}}{j\omega_0 L} = \frac{I_s R}{\omega_0 L}\angle -90° = Q_p \cdot I_s\angle -90°$$

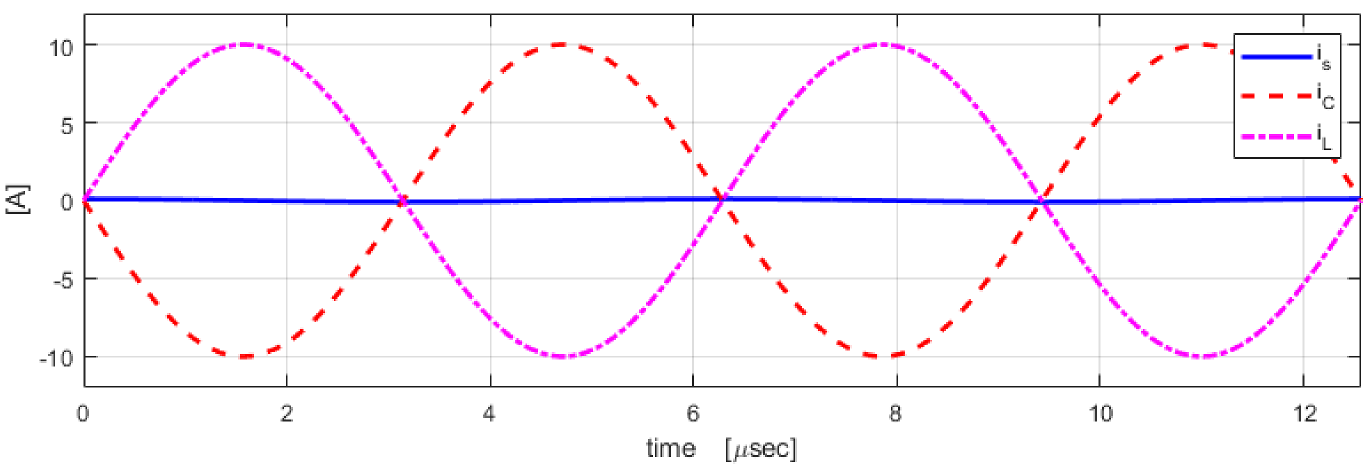
$$\mathbf{I_L} = 100 \cdot 100\ mA\angle -90°$$

$$\mathbf{I_L} = 10\ A\angle -90°$$

Parallel RLC Circuit – Voltages and Currents

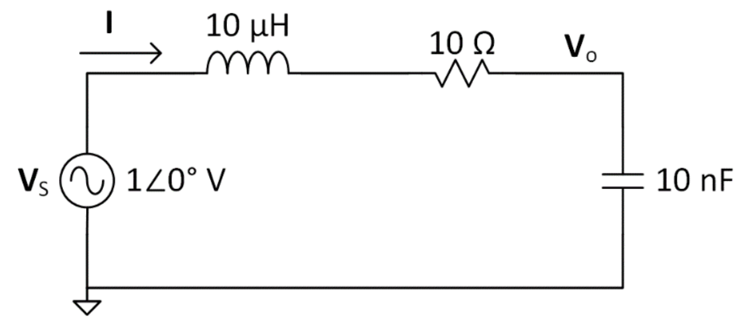


- $|\mathbf{I_S}| = 1\ V$
- $\angle \mathbf{V_o} = 0°$
- $|\mathbf{I_C}| = |\mathbf{I_L}| = Q_p|\mathbf{I_S}| = 10\ A$
- $|\mathbf{I_C}|$ and $|\mathbf{I_L}|$ are $180°$ out of phase
  - They cancel
  - KCL is satisfied

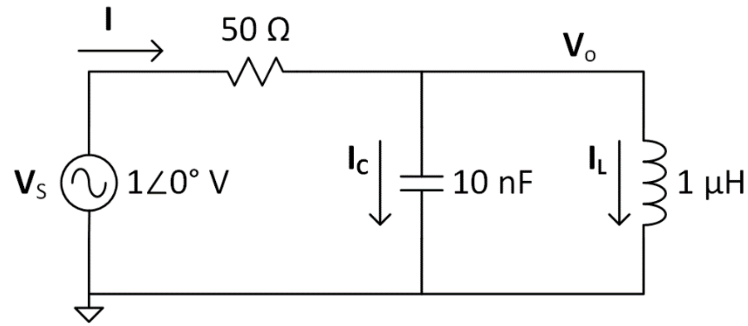


Example Problems

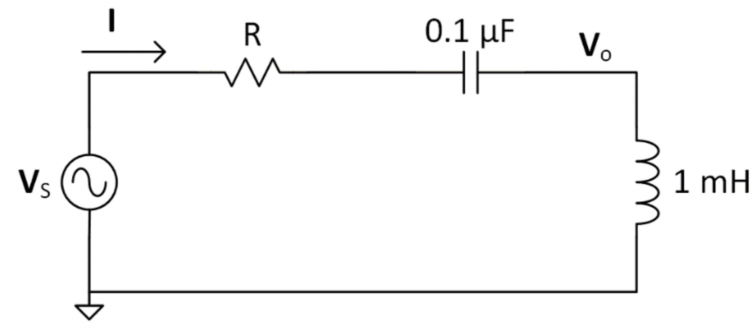
Determine the voltage across the capacitor, $\mathbf{V_o}$, at the resonant frequency.



Determine $\mathbf{V_o}$, $\mathbf{I}$, $\mathbf{I_C}$, and $\mathbf{I_L}$ at the resonant frequency.



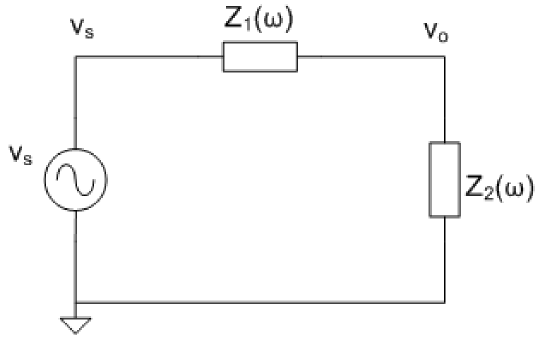
Determine R, such that $|\mathbf{V_o}| = 100|\mathbf{V_s}|$ at the resonant frequency.



Second-Order Filters

Second-Order Filters as Voltage Dividers

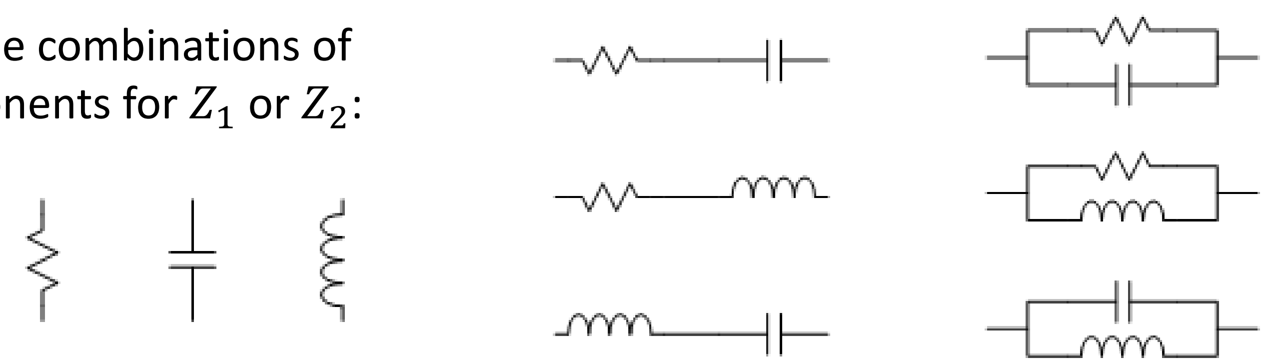
- Derive the frequency response functions of second-order filters by treating the circuits as voltage dividers



$$H(\omega) = \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)}$$

- Now, Z_1 and Z_2 can be either a single R, L, or C, or a series or parallel combination of any two

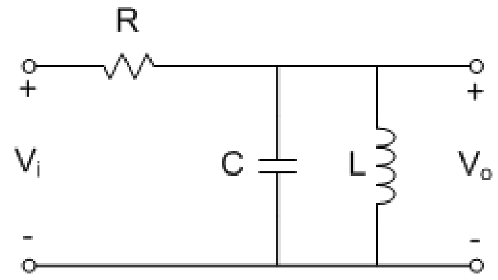
Possible combinations of components for Z_1 or Z_2:



Second-Order Band Pass Filter

Second-Order Band Pass Filter

- One option for a second-order band pass filter:
  - The frequency response function:



$$H(\omega) = \frac{Z_2}{Z_1 + Z_2}$$

where

$$Z_1 = R \quad \text{and} \quad Z_2 = \left[j\omega C + \frac{1}{j\omega L}\right]^{-1} = \frac{j\omega L}{1+(j\omega)^2 LC}$$

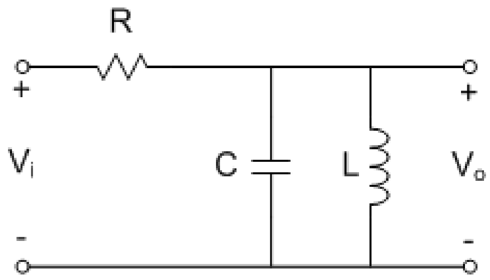
so

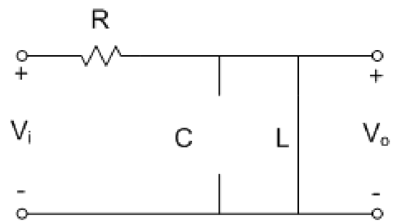
$$H(\omega) = \frac{\dfrac{j\omega L}{1+(j\omega)^2 LC}}{R + \dfrac{j\omega L}{1+(j\omega)^2 LC}} = \frac{j\omega L}{(j\omega)^2 RLC + j\omega L + R}$$

$$\boxed{H(\omega) = \frac{j\omega/RC}{(j\omega)^2 + j\omega/RC + 1/LC}}$$

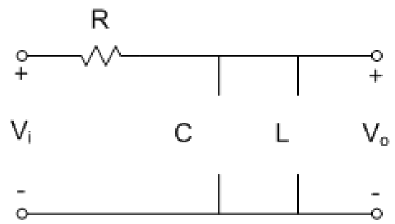
Second-Order Band Pass Filter

- Consider the filter's behavior at three limiting cases for frequency

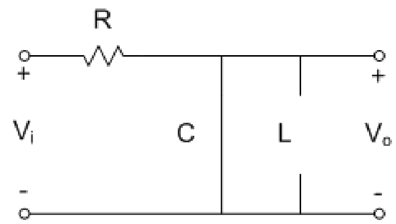




- $f \rightarrow 0$:
  - $C \rightarrow$ open
  - $L \rightarrow$ short
  - v_o shorted to ground
  - Gain $\rightarrow 0$



- $f = f_0$:
  - B_C, B_L cancel
  - $L||C \rightarrow$ open
  - $v_o = v_i$
  - Gain $\rightarrow 1$



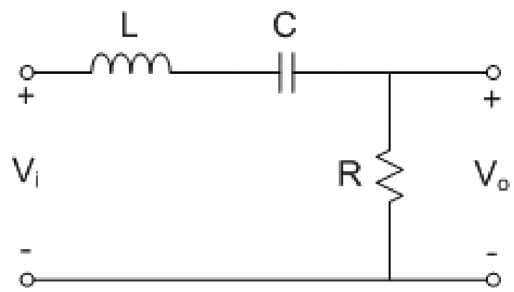
- $f \rightarrow \infty$:
  - $C \rightarrow$ short
  - $L \rightarrow$ open
  - v_o shorted to ground
  - Gain $\rightarrow 0$

Second-Order Band Pass Filter

- A second option for a second-order band pass filter:
  - Now, the impedances are:

$$Z_1 = j\omega L + \frac{1}{j\omega C} = \frac{(j\omega)^2 LC + 1}{j\omega C}$$

$$Z_2 = R$$



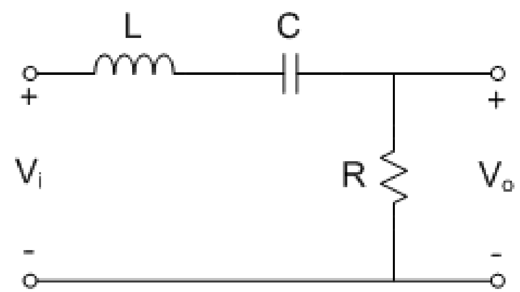
  - The frequency response function:

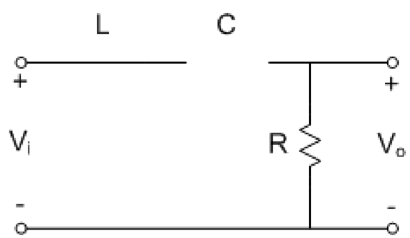
$$H(\omega) = \frac{R}{R + \frac{(j\omega)^2 LC + 1}{j\omega C}} = \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1}$$

$$\boxed{H(\omega) = \frac{j\omega R/L}{(j\omega)^2 + j\omega R/L + 1/LC}}$$

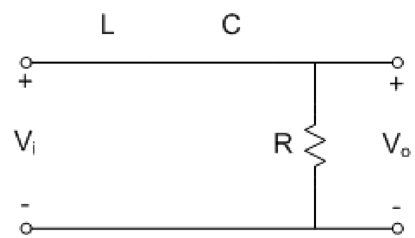
Second-Order Band Pass Filter

- Consider the filter's behavior at three limiting cases for frequency

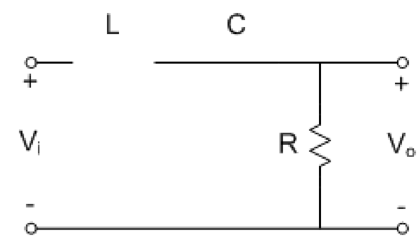




- $f \to 0$:
  - $C \to$ open
  - $L \to$ short
  - Current $\to 0$
  - $v_o \to 0$
  - Gain $\to 0$



- $f = f_0$:
  - X_C, X_L cancel
  - $L, C \to$ short
  - $v_o = v_i$
  - Gain $\to 1$



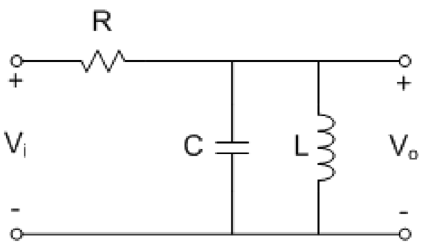
- $f \to \infty$:
  - $L \to$ open
  - $C \to$ short
  - Current $\to 0$
  - $v_o \to 0$
  - Gain $\to 0$

2^{nd}-Order BPF – General-Form Frequency Response

- Each of the two BPF variations has the same resonant frequency:

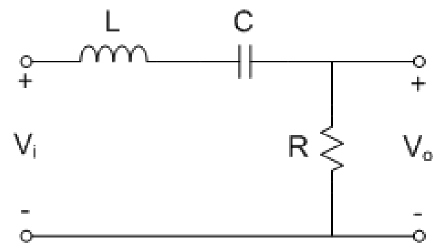
$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

- They have different frequency response functions and quality factors:



$$Q = \omega_0 RC = \frac{R}{\omega_0 L}$$

$$H(\omega) = \frac{j\omega/RC}{(j\omega)^2 + j\omega/RC + 1/LC}$$



$$Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}$$

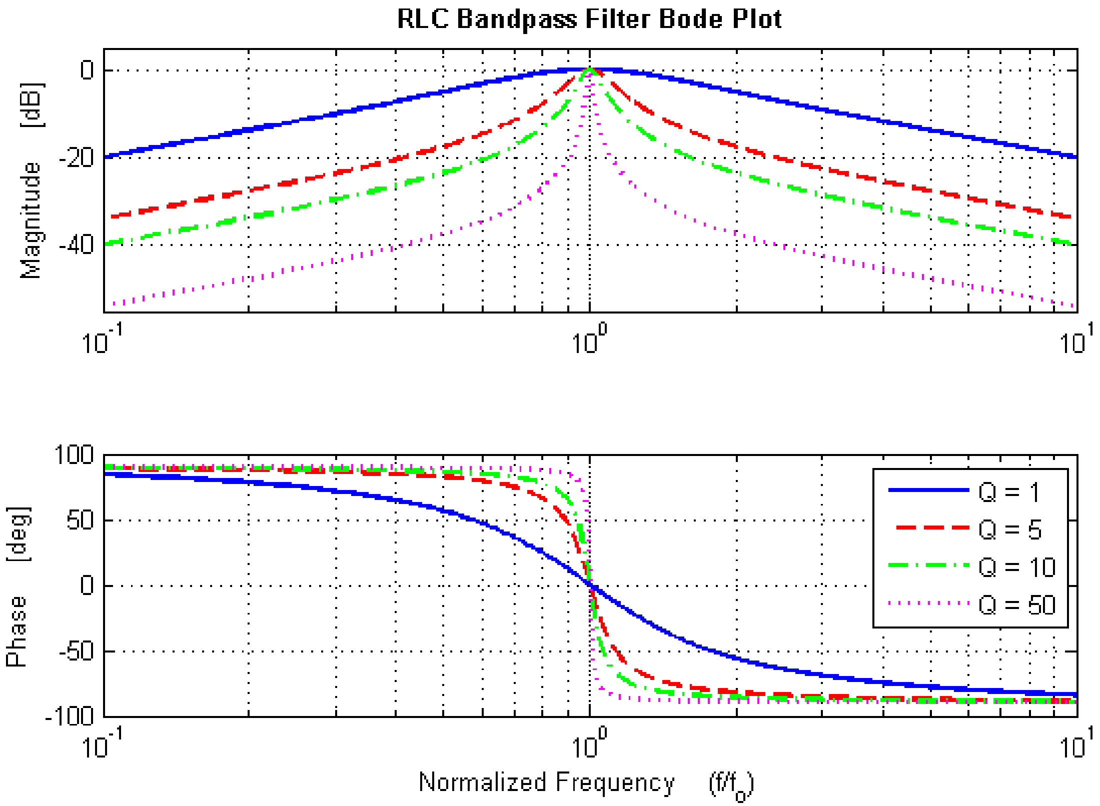
$$H(\omega) = \frac{j\omega R/L}{(j\omega)^2 + j\omega R/L + 1/LC}$$

- Each frequency response function can be expressed in terms of ω_0 and Q:

$$\boxed{H(\omega) = \frac{\frac{\omega_0}{Q} j\omega}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}}$$

Second-Order Band Pass Filter

- Same frequency response for each band pass filter



- Q determines the sharpness of the resonance
- ***Higher Q*** provides ***higher selectivity***
  - Narrower pass band
  - Steeper transition to the stop bands

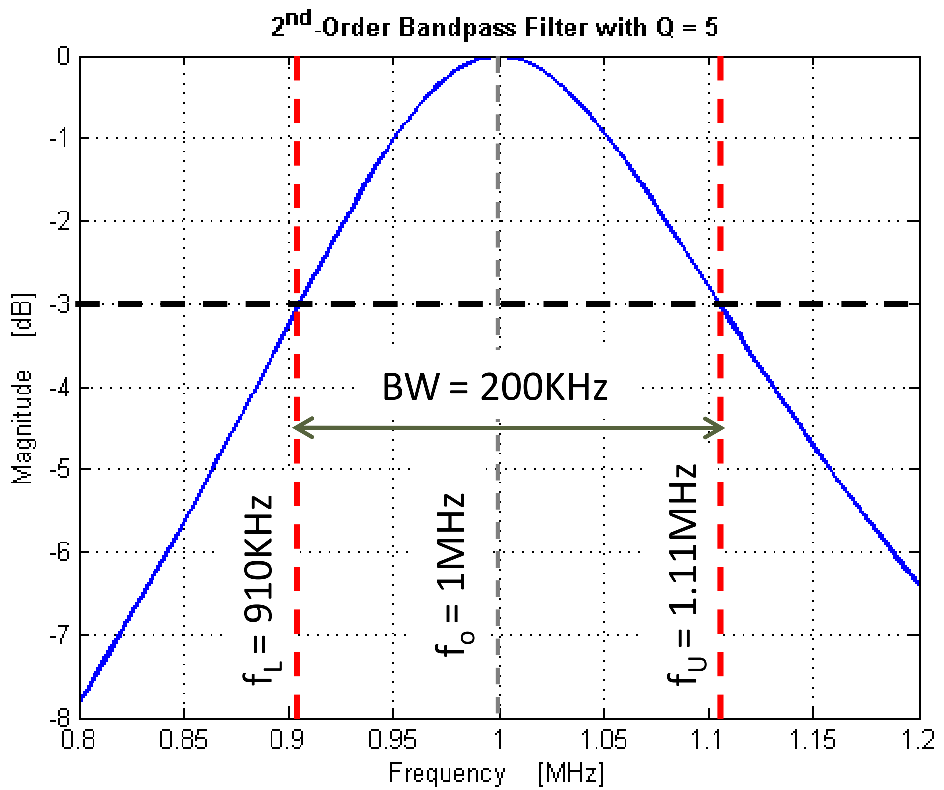
2nd-Order BP Filter – Bandwidth

- Bandwidth of a low pass filter is the 3 dB frequency
- A ***band pass filter*** has two 3 dB frequencies
  - ***Bandwidth*** is the difference between the two 3 dB frequencies

$$BW = f_U - f_L$$

  - Bandwidth is inversely proportional to Q

$$\boxed{BW = \frac{f_0}{Q}}$$

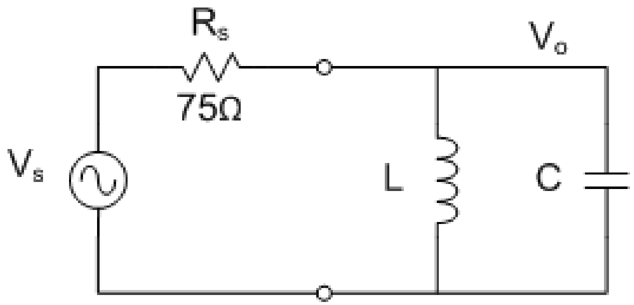


$$BW = f_U - f_L = \frac{f_0}{Q} = 200\ kHz$$

2^{nd}-Order BP Filter – Example

- Need a band pass filter to isolate a broadcast TV channel
  - Carrier frequency: 180MHz
  - Bandwidth of the filter: 6MHz
  - Thévenin equivalent resistance of signal source: 75Ω
- Use a parallel LC network
  - A ***tank circuit***

2^{nd}-Order BP Filter – Example



- Center frequency of the filter is:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 180\ MHz$$

- Specified bandwidth dictates the required Q value

$$Q = \frac{f_0}{BW} = \frac{180\ MHz}{6\ MHz} = 30$$

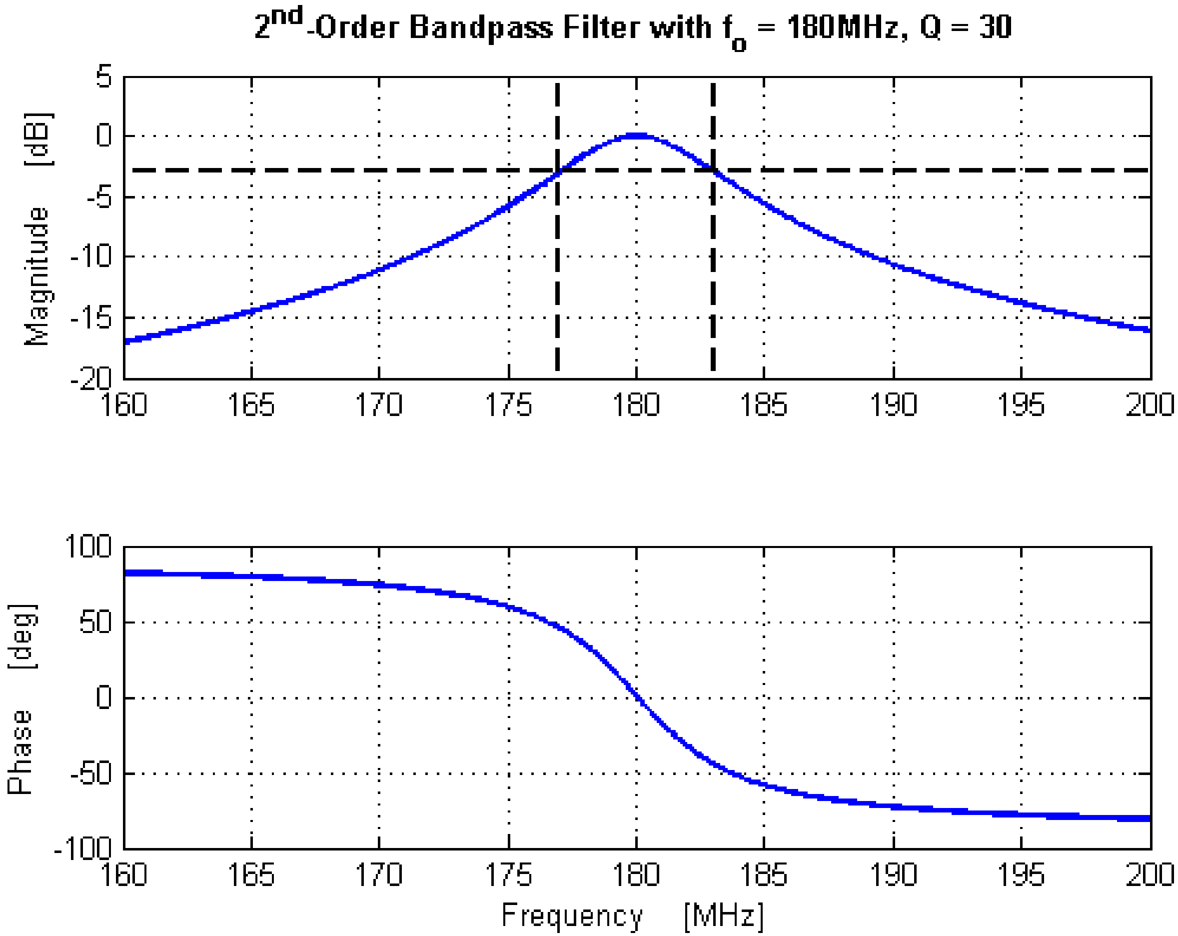
- Calculate the required inductance (and/or capacitance) using the values of R_s, Q, and f_0:

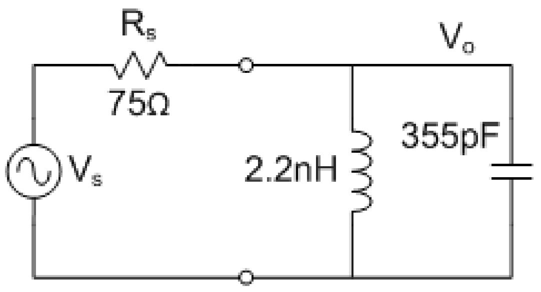
$$L = \frac{R_s}{\omega_0 Q} = \frac{75\ \Omega}{2\pi \cdot 180\ MHz \cdot 30} = 2.2\ nH$$

- Use the center frequency to determine the required capacitance

$$C = \frac{1}{L\omega_0^2} = \frac{1}{2.2\ nH(2\pi \cdot 180\ MHz)^2} = 355\ pF$$

2^{nd}-Order BP Filter – Example

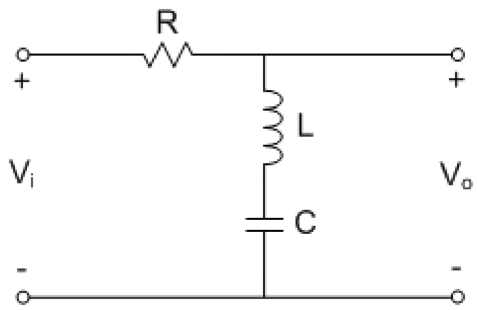




Second-Order Band Stop Filter

Second-Order Band Stop Filter

- One option for a second-order ***band stop***, or ***notch***, filter:
  - The frequency response function:



$$H(\omega) = \frac{Z_2}{Z_1 + Z_2}$$

where

$$Z_1 = R \qquad \text{and} \qquad Z_2 = j\omega L + \frac{1}{j\omega C} = \frac{(j\omega)^2 LC + 1}{j\omega C}$$

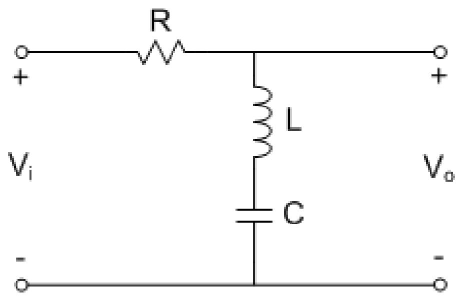
so

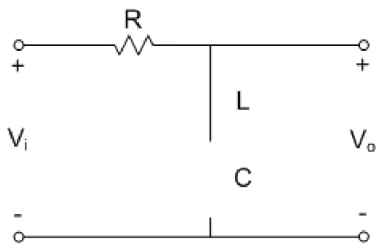
$$H(\omega) = \frac{\dfrac{(j\omega)^2 LC + 1}{j\omega C}}{R + \dfrac{(j\omega)^2 LC + 1}{j\omega C}} = \frac{(j\omega)^2 LC + 1}{(j\omega)^2 LC + j\omega RC + 1}$$

$$\boxed{H(\omega) = \frac{(j\omega)^2 + 1/LC}{(j\omega)^2 + j\omega R/L + 1/LC}}$$

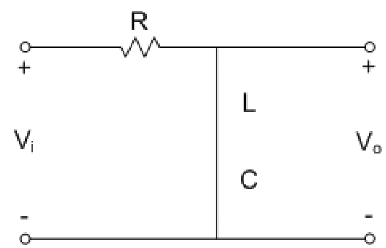
Second-Order Band Stop Filter

- Consider the filter's behavior at three limiting cases for frequency

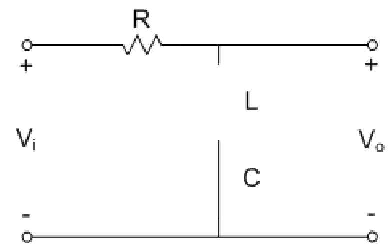




- $f \rightarrow 0$:
  - $C \rightarrow$ open
  - $L \rightarrow$ short
  - Current $\rightarrow 0$
  - $v_o \rightarrow v_i$
  - Gain $\rightarrow 1$



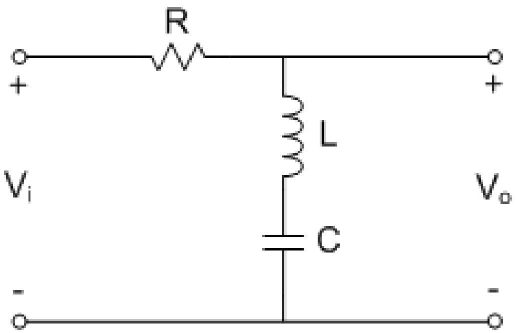
- $f = f_0$:
  - X_C, X_L cancel
  - $L,\ C \rightarrow$ short
  - v_o shorted to ground
  - Gain $\rightarrow 0$



- $f \rightarrow \infty$:
  - $C \rightarrow$ short
  - $L \rightarrow$ open
  - Current $\rightarrow 0$
  - $v_o \rightarrow v_i$
  - Gain $\rightarrow 1$

2^{nd}-Order BSF – General-Form Frequency Response

- ***Second-order band stop filter***
  - Resonant (center) frequency:



$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

  - Quality factor:

$$Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}$$

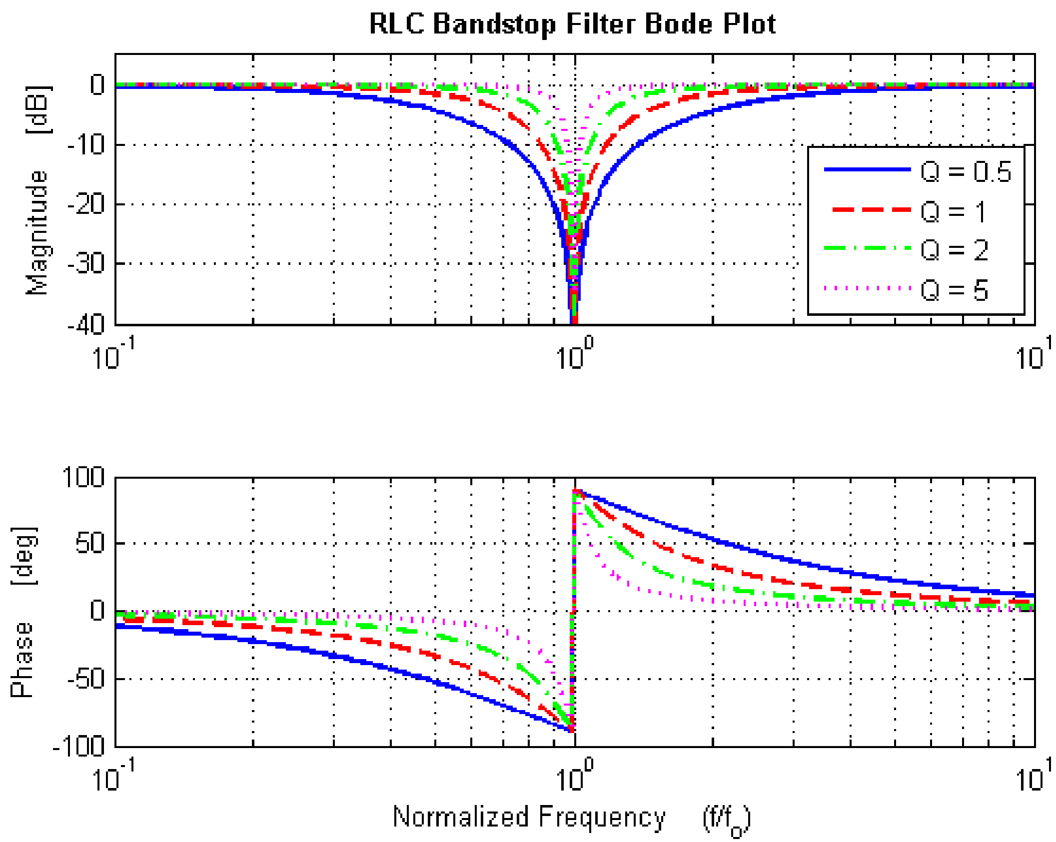
  - Frequency response function:

$$H(\omega) = \frac{(j\omega)^2 + 1/LC}{(j\omega)^2 + j\omega R/L + 1/LC}$$

  - General form, in terms of ω_0 and Q:

$$H(\omega) = \frac{(j\omega)^2 + \omega_0^2}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}$$

Second-Order Band Stop Filter



- All second-order notch filters provide same response as a function of Q and ω_0
- Q determines the sharpness of the response
- ***Higher Q*** provides ***higher selectivity***
  - Narrower stop band
  - Steeper transition to the pass bands

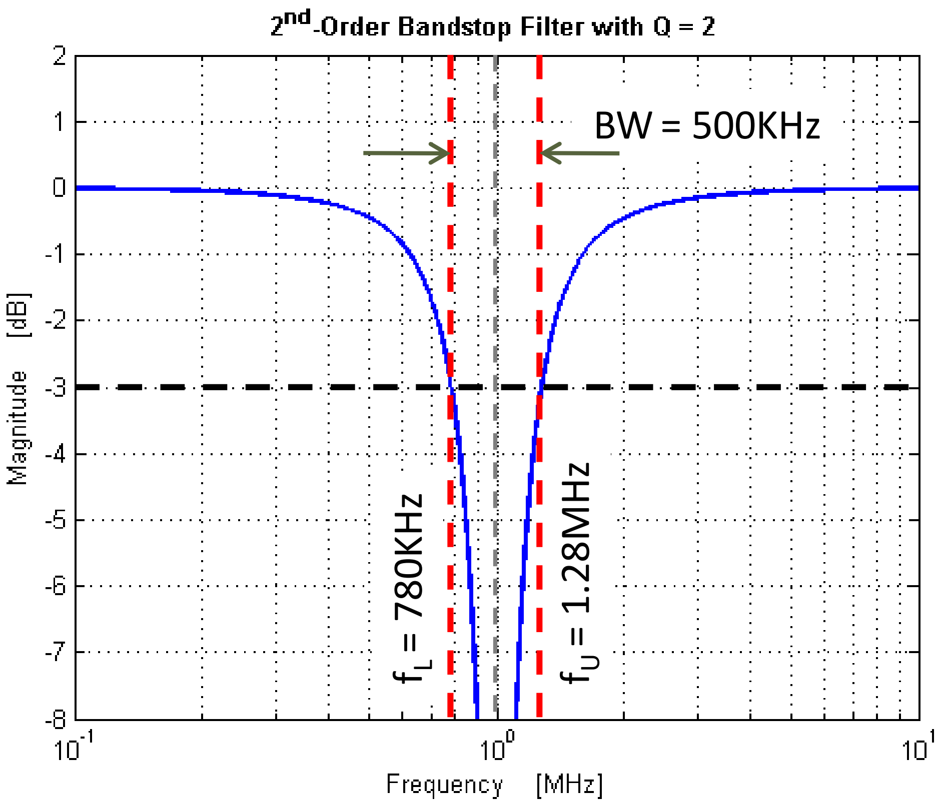
2nd-Order Notch Filter – Bandwidth

- Like the band pass filter, the ***band stop filter*** has two 3 dB frequencies
  - ***Bandwidth*** is the difference between the two 3 dB frequencies

$$BW = f_U - f_L$$

  - Bandwidth is inversely proportional to Q

$$\boxed{BW = \frac{f_0}{Q}}$$

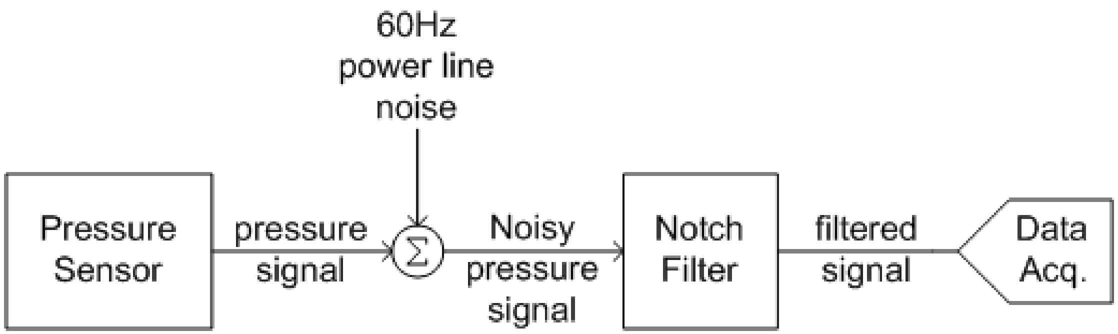


$$BW = f_U - f_L = \frac{f_0}{Q} = 500\ kHz$$

2nd-Order Notch Filter – Example

- Consider the following scenario:
  - Measuring transient pressure fluctuations inside an enclosed chamber
  - Pressure transducer monitored by a data acquisition system
  - Measured signal is small – all frequency content lies in the 1KHz – 15KHz range
  - Also interested in the average (DC) pressure value
    - AC coupling (HP filter) is not an option
    - Need to keep DC as well as 1KHz – 15KHz
  - Measurements are extremely noisy
    - Signal is completely buried in 60Hz power line noise
  - Design a notch filter to reject any 60Hz power line noise

2^{nd}-Order Notch Filter – Example



- Filter design considerations
  - Center frequency: 60 Hz
  - Attenuate signal of interest as little as possible
    - Set upper 3 dB frequency one decade below the lower end of the signal range (1 kHz)
  - Sensor output resistance: 100 Ω
  - DAQ system input resistance: 1 MΩ

2nd-Order Notch Filter – Example

- Upper 3 dB frequency is one decade below 1 kHz

$$f_U = 100\ Hz$$

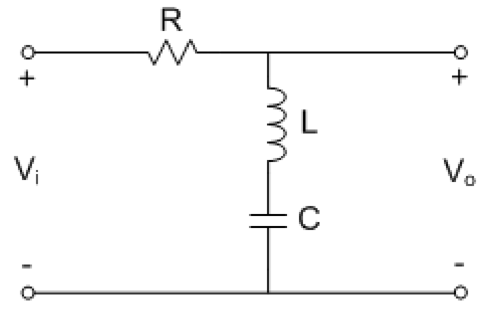
- Simplify by assuming that the 3 dB frequencies are evenly spaced about f_0

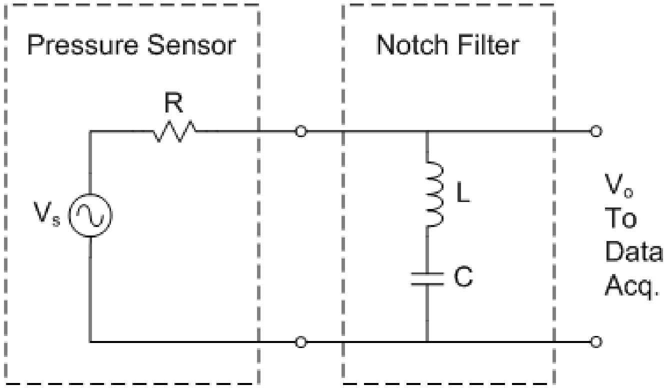
$$BW = 2(f_U - f_0) = 80\ Hz$$

- Required Q is then

$$Q = \frac{f_0}{BW} = \frac{60\ Hz}{80\ Hz} = 0.75$$

- Sensor output resistance can serve as the filter resistor
- DAQ input resistance of 1 MΩ is large enough to be neglected





2nd-Order Notch Filter – Example

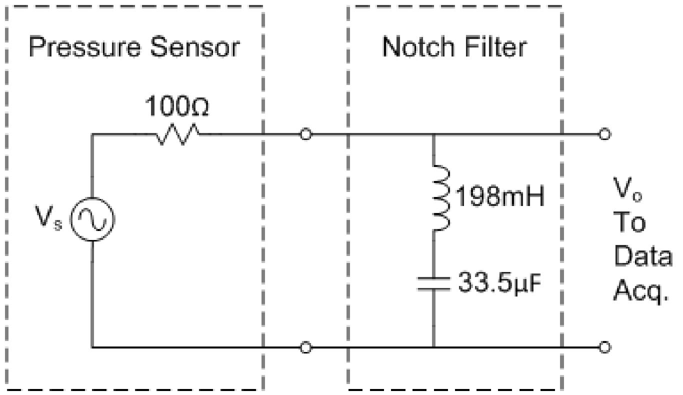
- Determine L and C values to satisfy f_0 and Q requirements
  - The required inductance:

$$L = \frac{Q \cdot R}{\omega_0} = \frac{0.75 \cdot 100\ \Omega}{2\pi \cdot 60\ Hz} = 198\ mH$$

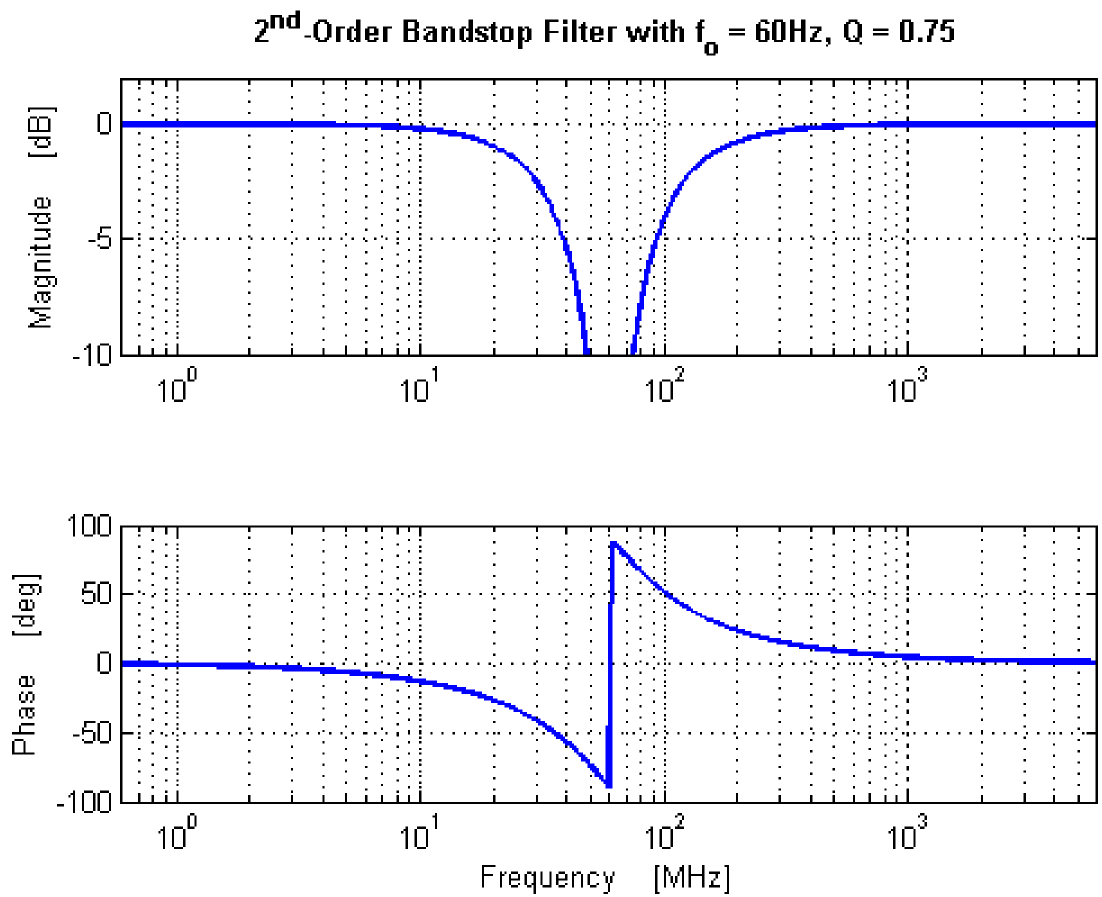
  - Calculate C to place the center frequency at 60 Hz

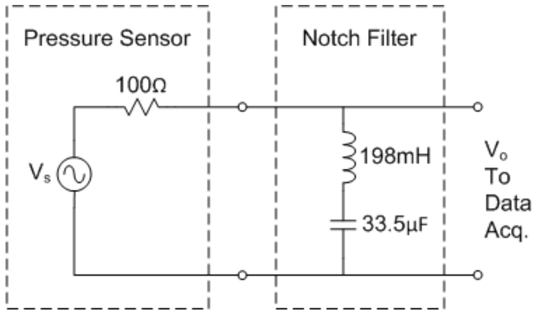
$$C = \frac{1}{\omega_0^2 L} = \frac{1}{(2\pi \cdot 60\ Hz)^2 \cdot 198\ mH} = 35.5\ \mu F$$

- A couple things worth noting:
  - Some iteration selecting standard-value components would be required
  - Accuracy and stability of sensor output resistance would need to be verified



2nd-Order Notch Filter – Example

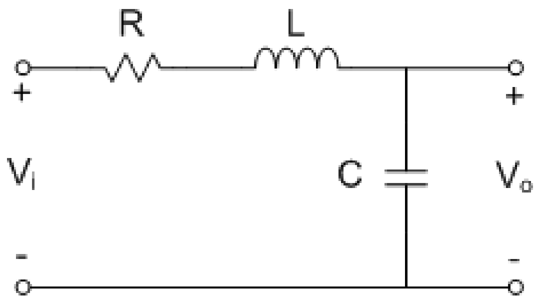




Second-Order Low Pass Filter

Second-Order Low Pass Filter

- Second-order low pass filter:
  - The frequency response function:



$$H(\omega) = \frac{Z_2}{Z_1 + Z_2}$$

where

$$Z_1 = R + j\omega L \quad \text{and} \quad Z_2 = \frac{1}{j\omega C}$$

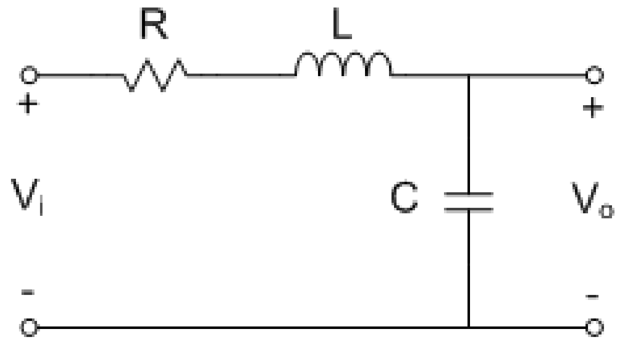
so

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{(j\omega)^2 LC + j\omega RC + 1}$$

$$\boxed{H(\omega) = \frac{1/LC}{(j\omega)^2 + j\omega R/L + 1/LC}}$$

2nd-Order LPF – General-Form Frequency Response

- ***Second-order low pass filter***



  - Resonant frequency:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

  - Quality factor:

$$Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}$$

  - Frequency response function:

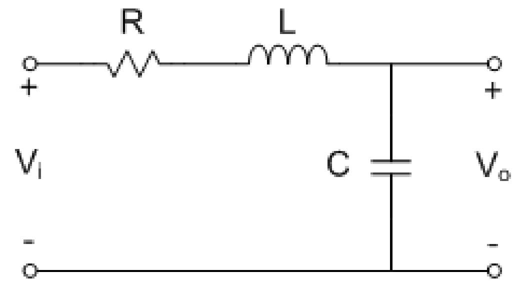
$$H(\omega) = \frac{1/LC}{(j\omega)^2 + j\omega R/L + 1/LC}$$

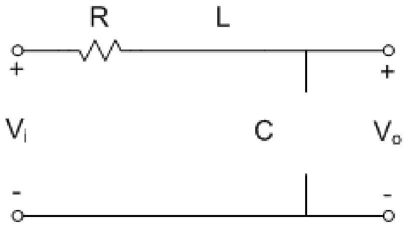
  - General form, in terms of ω_0 and Q:

$$H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}$$

Second-Order Low Pass Filter

- Consider the filter's behavior at three limiting cases for frequency

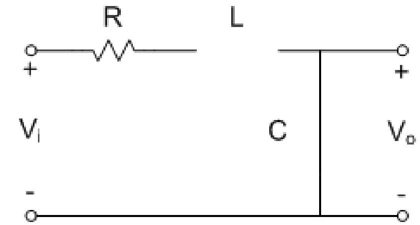




- $f \to 0$:
  - $L \to$ short
  - $C \to$ open
  - Current $\to 0$
  - $v_o \to v_i$
  - Gain $\to 1$

?

- $f = f_0$:
  - Behavior at resonance is a bit trickier here



- $f \to \infty$:
  - $L \to$ open
  - $C \to$ short
  - v_o shorted to ground
  - Gain $\to 0$

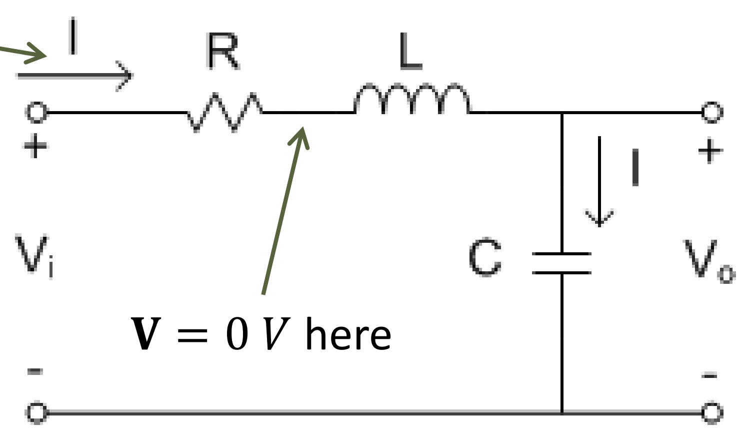
Second-Order LPF at Resonance

- Input impedance at resonance: $Z_{in}(\omega_0) = R$
- The series LC section is essentially a short
  - But, neither the inductor nor the capacitor, individually, are shorts
  - And, output is taken across the capacitor
    - Recall that at resonance, capacitor and inductor voltages can exceed the input voltage

---

$Z_{in} = R$, so

$$\mathbf{I} = \frac{\mathbf{V_i}}{R}$$



$\mathbf{V} = 0\ V$ here

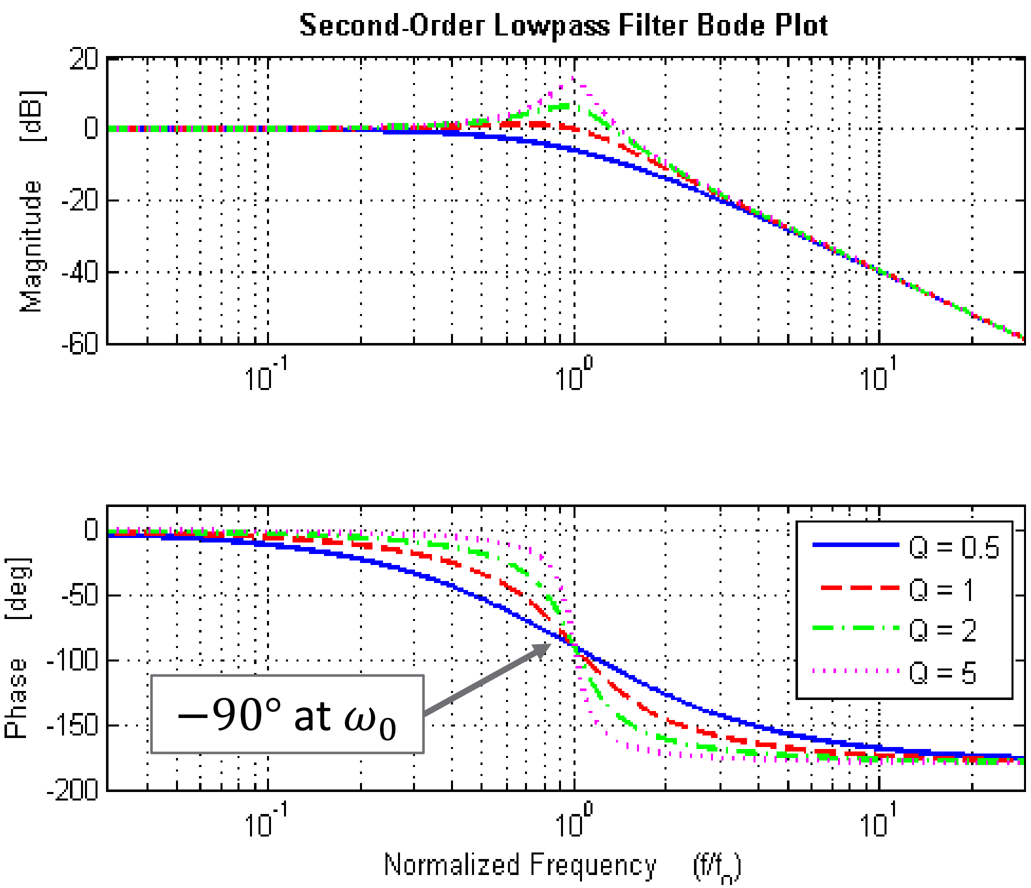
Output voltage phasor is:

$$\mathbf{V_o} = \frac{\mathbf{I}}{j\omega_0 C}$$

$$\mathbf{V_o} = \frac{\mathbf{V_i}}{\omega_0 RC} \angle -90°$$

which may be larger than $\mathbf{V_i}$

Second-Order Low Pass Filter



- Second-order roll-off rate:

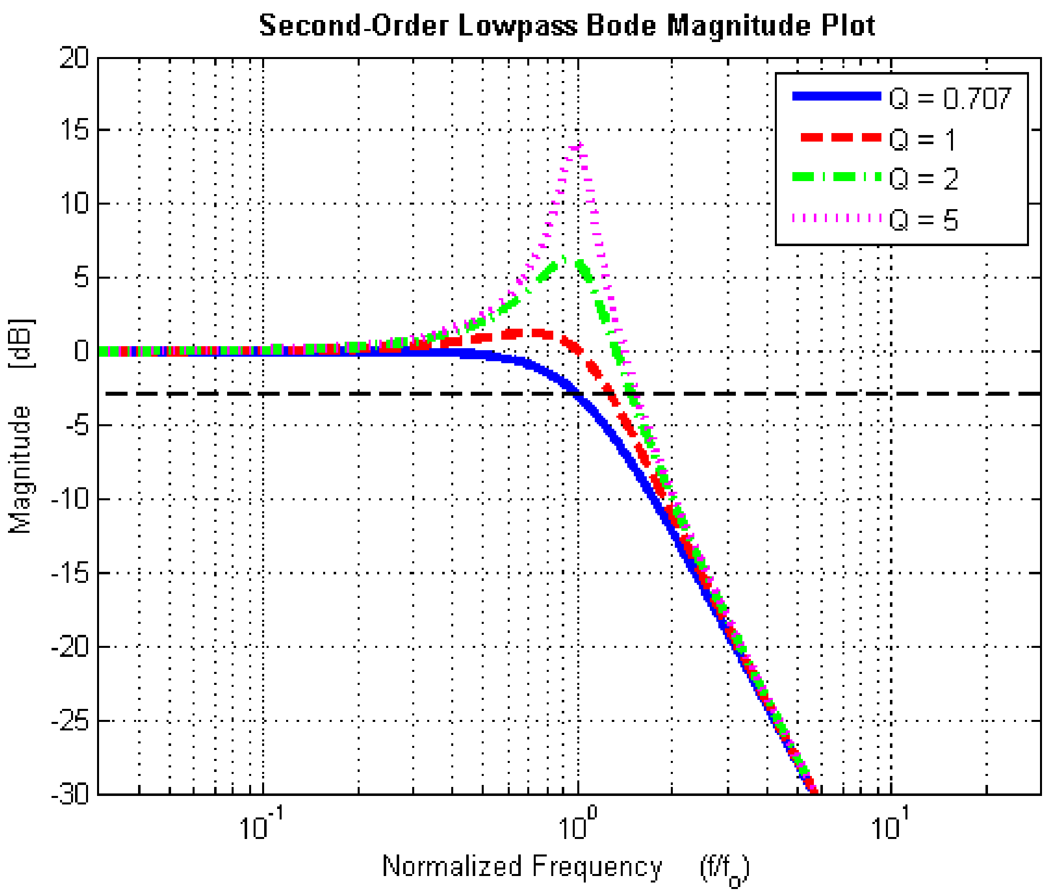
$$40\,\frac{dB}{dec}$$

- Q determines:
  - Amount of peaking
  - Rate of phase transition
- No peaking at all for

$$Q \leq \frac{1}{\sqrt{2}} = 0.707$$

- Phase at ω_0: $-90°$

2^{nd}-Order Low Pass Filter – 3dB Bandwidth



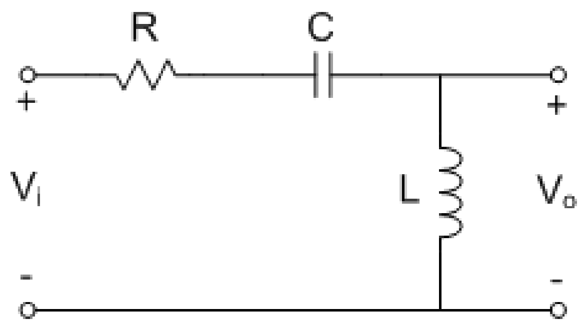
- 3 dB bandwidth is a function of Q
  - Increases with Q
- Peaking occurs for $Q > 0.707$
- For $Q = 0.707$
  - ***Maximally-flat response***
  - ***Butterworth response***
  - The 3 dB frequency is equal to the resonant frequency

$$f_c = f_0$$

Second-Order High Pass Filter

Second-Order High Pass Filter

- Second-order high pass filter:
  - The frequency response function:



$$H(\omega)=\frac{Z_2}{Z_1+Z_2}$$

where

$$Z_1=R+\frac{1}{j\omega C}\quad \text{and} \quad Z_2=j\omega L$$

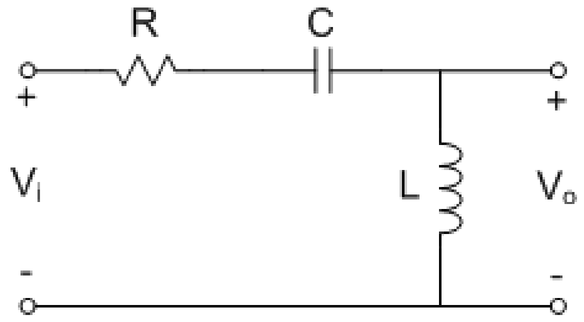
so

$$H(\omega)=\frac{j\omega L}{R+\frac{1}{j\omega C}+j\omega L}=\frac{(j\omega)^2LC}{(j\omega)^2LC+j\omega RC+1}$$

$$\boxed{H(\omega)=\frac{(j\omega)^2}{(j\omega)^2+j\omega R/L+1/LC}}$$

2^{nd}-Order HPF – General-Form Frequency Response

- ***Second-order high pass filter***



  - Resonant frequency:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

  - Quality factor:

$$Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}$$

  - Frequency response function:

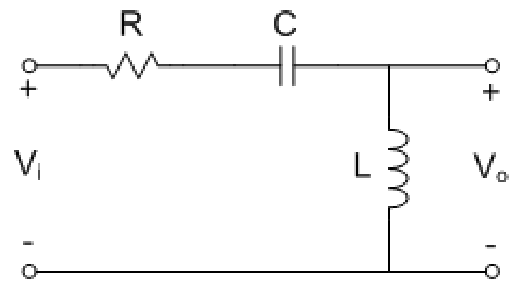
$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + j\omega R/L + 1/LC}$$

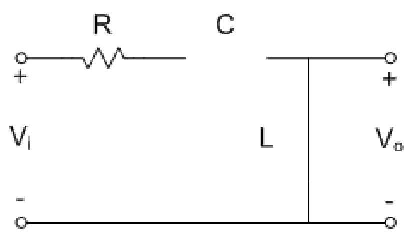
  - General form, in terms of ω_0 and Q:

$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}$$

Second-Order High Pass Filter

- Consider the filter's behavior at three limiting cases for frequency

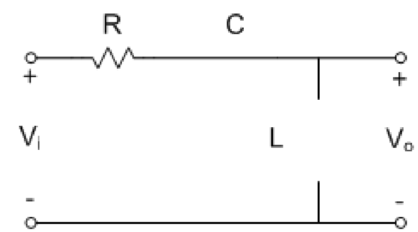




- $f \rightarrow 0$:
  - $L \rightarrow$ short
  - $C \rightarrow$ open
  - v_o shorted to ground
  - Gain $\rightarrow 0$

?

- $f = f_0$:
  - Behavior at resonance is, once again, a bit more complicated



- $f \rightarrow \infty$:
  - $L \rightarrow$ open
  - $C \rightarrow$ short
  - Current $\rightarrow 0$
  - $v_o \rightarrow v_i$
  - Gain $\rightarrow 1$

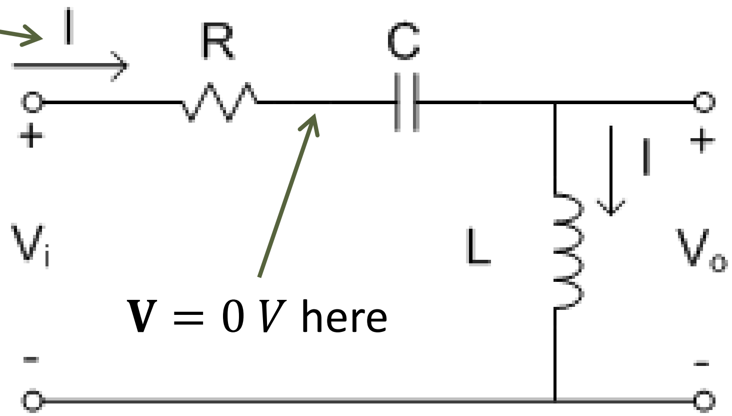
Second-Order HPF at Resonance

- Input impedance at resonance: $Z_{in}(\omega_0) = R$
- The series LC section is essentially a short
  - But, neither the inductor nor the capacitor, individually, are shorts
  - And, output is taken across the inductor
    - Recall that at resonance, capacitor and inductor voltages can exceed the input voltage

---

$Z_{in} = R$, so

$$\mathbf{I} = \frac{\mathbf{V_i}}{R}$$



$\mathbf{V} = 0\ V$ here

Output voltage phasor is:

$$\mathbf{V_o} = \mathbf{I} \cdot j\omega_0 L$$

$$\mathbf{V_o} = \frac{\mathbf{V_i}}{R}\omega_0 L\angle 90°$$

which may be larger than $\mathbf{V_i}$

Second-Order High Pass Filter

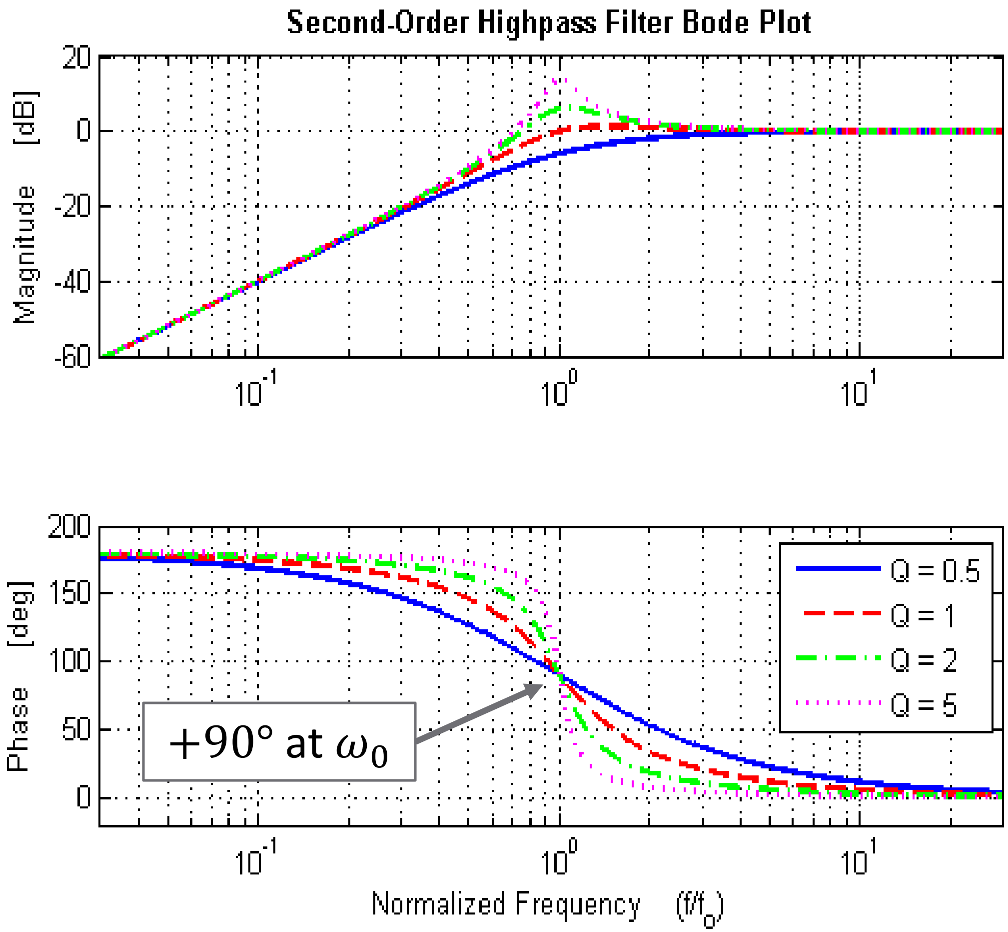
- Second-order roll-off rate:

$$40\,\frac{dB}{dec}$$

- Q determines:
  - Amount of peaking
  - Rate of phase transition
- No peaking at all for

$$Q \le \frac{1}{\sqrt{2}} = 0.707$$

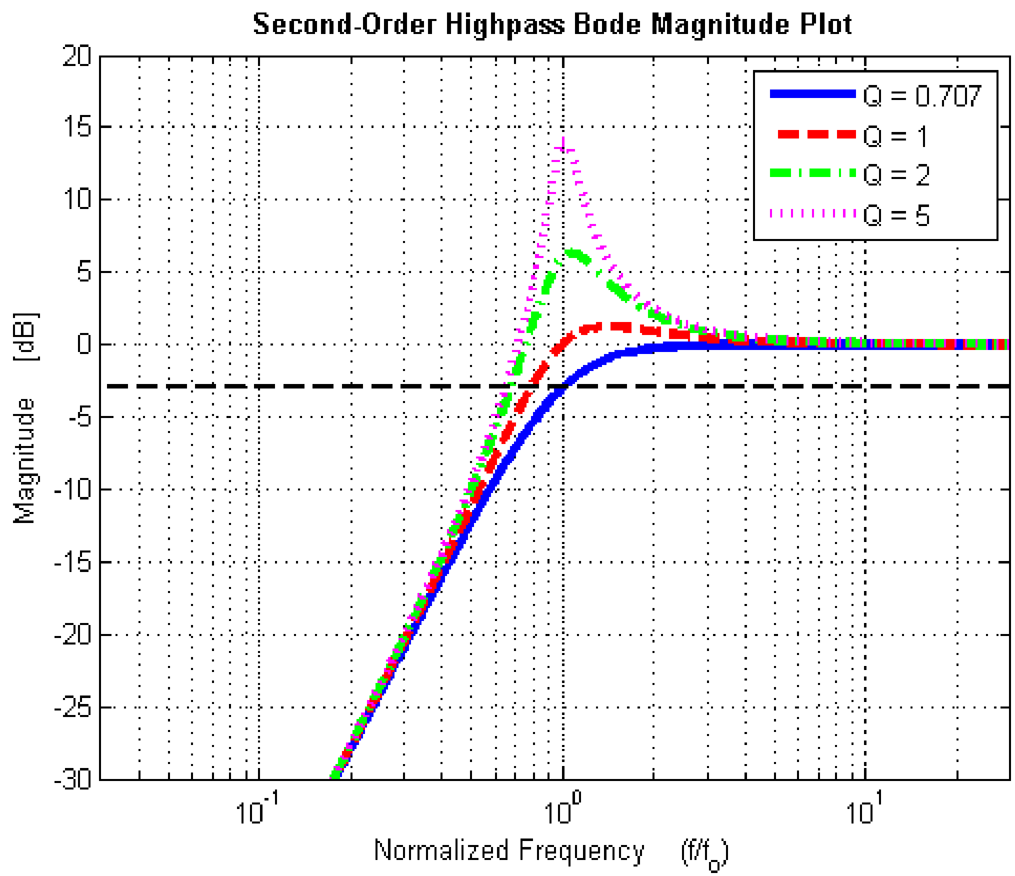
- Phase at ω_0: $+90°$



2nd-Order High Pass Filter – 3dB Bandwidth

- Corner frequency is a function of Q
  - Decreases with increasing Q
- Peaking occurs for $Q > 0.707$
- For $Q = 0.707$
  - ***Maximally-flat response***
  - ***Butterworth response***
  - The 3 dB frequency is equal to the resonant frequency

$$f_c = f_0$$



Damping Ratio - ζ

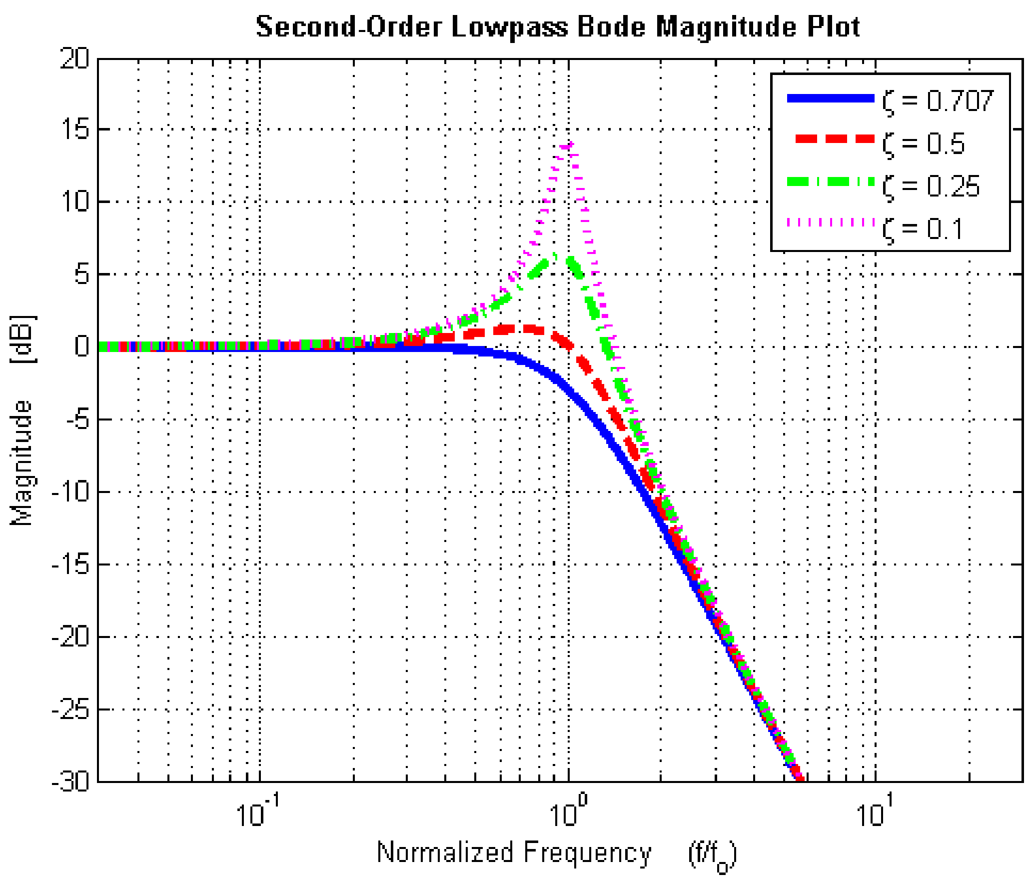
- We've been using ***quality factor*** to describe second-order filter response
  - A measure of the sharpness of the resonance
  - For band pass/stop filters, Q tells us about ***bandwidth***
  - For low/high pass filters, Q tells us about ***peaking***

- Another way to describe the same characteristic: ***damping ratio***, ζ
  - Damping ratio is inversely proportional to Q:

$$\boxed{\zeta = \frac{1}{2Q}}$$

  - A measure of the amount of ***damping*** in a circuit/system
  - Higher ζ implies a less resonant system
    - Less peaking
    - Wider bandwidth for band pass/stop filters

2nd-Order Low Pass Response vs. ζ



- As ζ goes down,
  - Less damping
  - More peaking
- No peaking at all for $\zeta \geq 0.707$
- For $\zeta = 0.707$
  - ***Maximally-flat response***
  - ***Butterworth response***
  - The 3 dB frequency is equal to the resonant frequency

$$f_c = f_0$$

General-Form Freq. Response Functions in Terms of ζ

Low pass in terms of Q:

$$H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}$$

Low pass in terms of ζ:

$$H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2}$$

High pass in terms of Q:

$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + \frac{\omega_0}{Q} j\omega + \omega_0^2}$$

High pass in terms of ζ:

$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2}$$

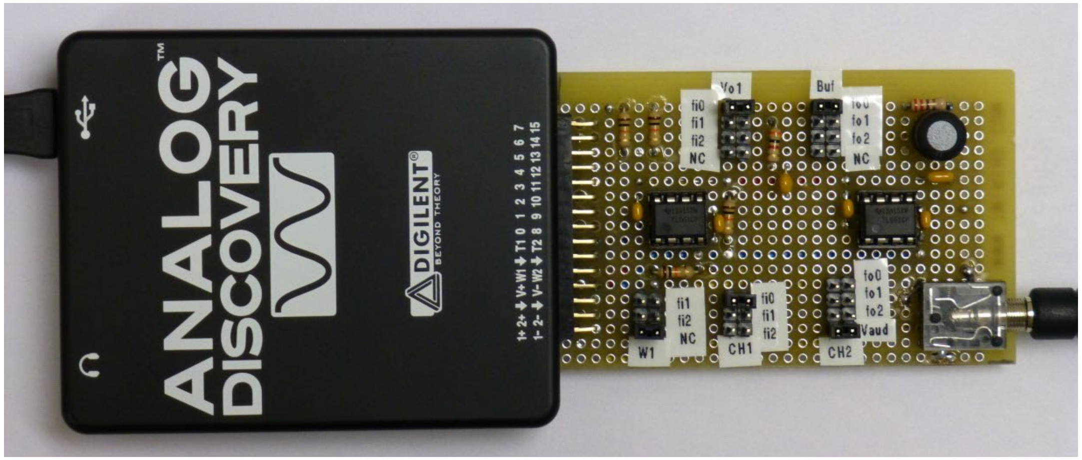
Audio Filter Demo

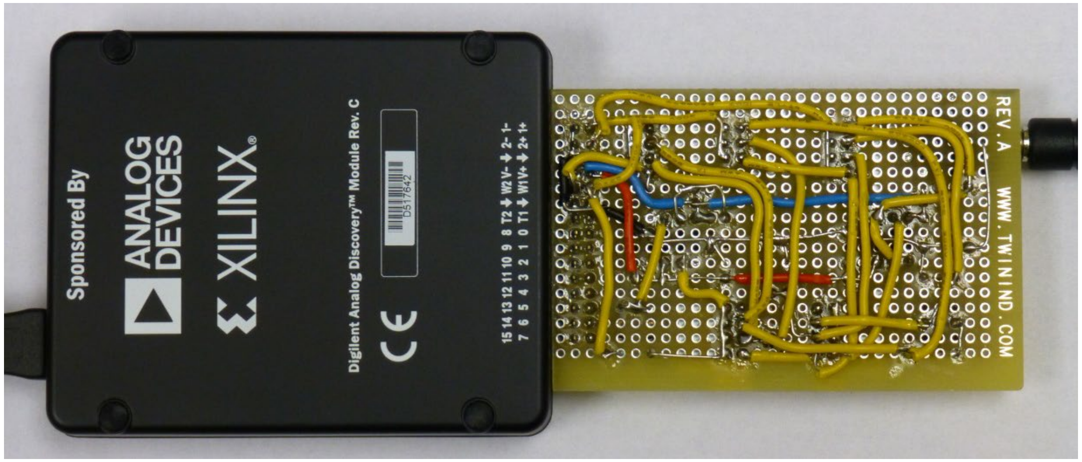
Analog Discovery Instrument

- 2-chan. Scope
  - 14-bit, 100MSa/s
  - 5MHz bandwidth
- 2-chan. function generator
  - 14-bit, 100MSa/s
  - 5MHz bandwidth
- 2-chan. spectrum analyzer
- Network analyzer
- Voltmeter
- ±5V power supplies
- 16-chan. logic analyzer
- 16-chan. digital pattern generator
- USB connectivity

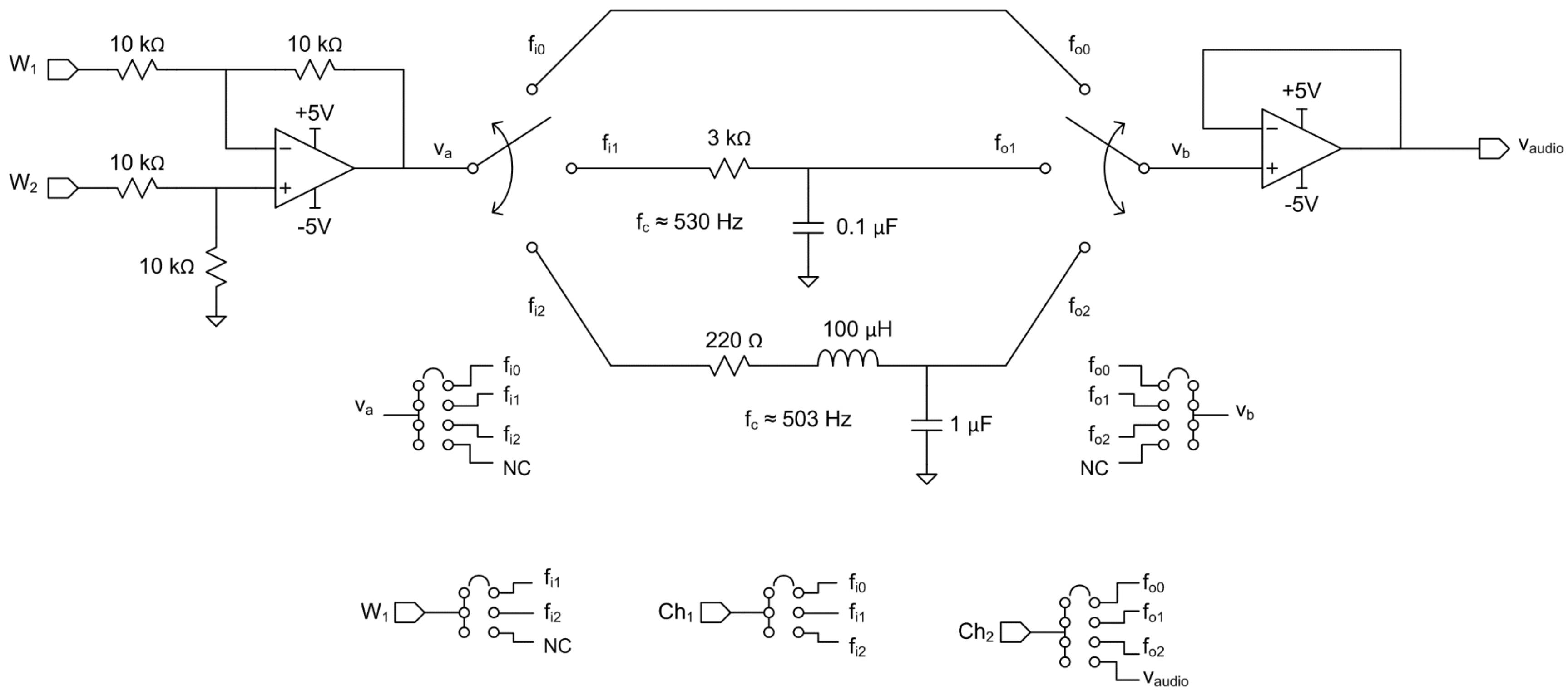
Analog Discovery – Audio Demo

- Demo board plugs in to Analog Discovery module
- Summation of multiple tones
- Optional filtering of audio signal
- 3.5 mm audio output jack





Analog Discovery – Audio Demo



Example Problems

The higher-frequency signal is unwanted noise. Design a second-order Butterworth LPF to attenuate the higher-frequency component by 40 dB.

What is the SNR at the output of the filter?