SECTION 7: THREE-PHASE CIRCUIT FUNDAMENTALS
Balanced Three-Phase Networks
Balanced Three-Phase Networks

- We are accustomed to *single-phase* power in our homes and offices
  - A single *line* voltage referenced to a *neutral*

- Electrical power is generated, transmitted, and largely consumed (by industrial customers) as *three-phase power*
  - Three individual line voltages and (possibly) a neutral
  - Line voltages all differ in phase by $\pm 120^\circ$
Δ- and Y-Connected Networks

- Two possible three-phase configurations
  - Applies to both sources and loads

Y-Connected Source

- Y-connected network has a neutral node
- Δ-connected network has no neutral
In the Y network, voltages $V_{an}$, $V_{bn}$, and $V_{cn}$ are **line-to-neutral voltages**.

A three-phase source is **balanced** if

- Line-to-neutral voltages have equal magnitudes
- Line-to-neutral voltage are each $120^\circ$ out of phase with one another

A three-phase network is balanced if

- Sources are balanced
- The impedances connected to each phase are equal
Line-to-Neutral Voltages

The line-to-neutral voltages are

\[
\begin{align*}
V_{an} &= V_{LN} \angle 0^\circ \\
V_{bn} &= V_{LN} \angle -120^\circ \\
V_{cn} &= V_{LN} \angle -240^\circ = V_{LN} \angle +120^\circ
\end{align*}
\]

This is a **positive-sequence** or **abc-sequence** source

- \(V_{an}\) leads \(V_{bn}\), which leads \(V_{cn}\)

Can also have a **negative-** or **acb-sequence** source

- \(V_{an}\) leads \(V_{cn}\), which leads \(V_{bn}\)

We’ll always assume **positive-sequence** sources
The voltages between the three phases are *line-to-line voltages*

Apply KVL to relate line-to-line voltages to line-to-neutral voltages

\[ V_{ab} - V_{an} + V_{bn} = 0 \]
\[ V_{ab} = V_{an} - V_{bn} \]

We know that
\[ V_{an} = V_{LN} \angle 0^\circ \]
and
\[ V_{bn} = V_{LN} \angle -120^\circ \]

so
\[ V_{ab} = V_{LN} \angle 0^\circ - V_{LN} \angle -120^\circ = V_{LN} \left(1 \angle 0^\circ - 1 \angle -120^\circ\right) \]
\[ V_{ab} = V_{LN} \left[1 - \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\right] = V_{LN} \left[\frac{3}{2} + j\frac{\sqrt{3}}{2}\right] \]
\[ V_{ab} = \sqrt{3}V_{LN} \angle 30^\circ \]
Again applying KVL, we can find \( V_{bc} \)

\[
V_{bc} = V_{bn} - V_{cn}
\]

\[
V_{bc} = V_{LN} \angle -120^\circ - V_{LN} \angle 120^\circ
\]

\[
V_{bc} = V_{LN} \left[ \left( -\frac{1}{2} - j\frac{\sqrt{3}}{2} \right) - \left( -\frac{1}{2} + j\frac{\sqrt{3}}{2} \right) \right]
\]

\[
V_{bc} = V_{LN} (-j\sqrt{3})
\]

\[
V_{bc} = \sqrt{3}V_{LN} \angle -90^\circ
\]

Similarly,

\[
V_{ca} = \sqrt{3}V_{LN} \angle 150^\circ
\]
The line-to-line voltages, with $V_{an}$ as the reference:

- $V_{ab} = \sqrt{3}V_{LN}\angle30^\circ$
- $V_{bc} = \sqrt{3}V_{LN}\angle-90^\circ$
- $V_{ca} = \sqrt{3}V_{LN}\angle150^\circ$

Line-to-line voltages are $\sqrt{3}$ times the line-to-neutral voltage.

Can also express in terms of individual line-to-neutral voltages:

- $V_{ab} = \sqrt{3}V_{an}\angle30^\circ$
- $V_{bc} = \sqrt{3}V_{bn}\angle30^\circ$
- $V_{ca} = \sqrt{3}V_{cn}\angle30^\circ$
Line Currents in Balanced 3φ Networks

- We can use the line-to-neutral voltages to determine the line currents
  - Y-connected source and load
  - Balanced load – all impedances are equal: $Z_Y$

\[
I_a = \frac{V_{AN}}{Z_Y} = \frac{V_{LN} \angle 0^\circ}{Z_Y} \\
I_b = \frac{V_{BN}}{Z_Y} = \frac{V_{LN} \angle -120^\circ}{Z_Y} \\
I_c = \frac{V_{CN}}{Z_Y} = \frac{V_{LN} \angle +120^\circ}{Z_Y}
\]

- Line currents are balanced as long as the source and load are balanced
Neutral Current in Balanced 3ϕ Networks

- Apply KCL to determine the neutral current

\[ I_n = I_a + I_b + I_c \]

\[ I_n = \frac{V_{LN}}{Z_Y} [1 \angle 0^\circ + 1 \angle -120^\circ + 1 \angle 120^\circ] \]

\[ I_n = \frac{V_{LN}}{Z_Y} \left[ 1 + \left( -\frac{1}{2} - j \frac{\sqrt{3}}{2} \right) + \left( -\frac{1}{2} + j \frac{\sqrt{3}}{2} \right) \right] \]

\[ I_n = 0 \]

- The neutral conductor carries no current in a balanced three-phase network
Delta- & Wye-Connected Networks
As for sources, three-phase loads can also be connected in two different configurations.

- **Y-Connected Load**
  - The Y load has a neutral connection, but the Δ load does not.

- **Δ-Connected Load**
  - Currents in a Y-connected load are the line currents we just determined.
  - Next, we’ll look at currents in a Δ-connected load.
Balanced $\Delta$-Connected Loads

- We can use line-to-line voltages to determine the currents in $\Delta$-connected loads.

\[
I_{AB} = \frac{V_{AB}}{Z_\Delta} = \frac{\sqrt{3}V_{AN} \angle 30^\circ}{Z_\Delta} = \frac{\sqrt{3}V_{LN} \angle 30^\circ}{Z_\Delta}
\]

\[
I_{BC} = \frac{V_{BC}}{Z_\Delta} = \frac{\sqrt{3}V_{BN} \angle 30^\circ}{Z_\Delta} = \frac{\sqrt{3}V_{LN} \angle -90^\circ}{Z_\Delta}
\]

\[
I_{CA} = \frac{V_{CA}}{Z_\Delta} = \frac{\sqrt{3}V_{CN} \angle 30^\circ}{Z_\Delta} = \frac{\sqrt{3}V_{LN} \angle 150^\circ}{Z_\Delta}
\]
Balanced Δ-Connected Loads

- Applying KCL, we can determine the line currents:

\[ I_a = I_{AB} - I_{CA} \]

\[ I_a = \frac{\sqrt{3}V_{LN}}{Z_\Delta} [1\angle 30^\circ - 1\angle 150^\circ] \]

\[ I_a = \frac{\sqrt{3}V_{LN}}{Z_\Delta} \left( \frac{\sqrt{3}}{2} + j\frac{1}{2} \right) - \left( -\frac{\sqrt{3}}{2} + j\frac{1}{2} \right) = \frac{\sqrt{3}V_{LN}}{Z_\Delta} [\sqrt{3}] = \frac{3V_{LN}}{Z_\Delta} \]

- The other line currents can be found similarly:

\[ I_a = \frac{3V_{LN} \angle 0^\circ}{Z_\Delta} = \sqrt{3}I_{AB} \angle -30^\circ \]

\[ I_b = \frac{3V_{LN} \angle -120^\circ}{Z_\Delta} = \sqrt{3}I_{BC} \angle -30^\circ \]

\[ I_c = \frac{3V_{LN} \angle 120^\circ}{Z_\Delta} = \sqrt{3}I_{CA} \angle -30^\circ \]
Delta-Y Conversion

- Analysis is often simpler when dealing with Y-connected loads
  - Would like a way to convert Δ loads to Y loads (and vice versa)

- For a Y load and a Δ load to be equivalent, they must result in equal line currents
Δ — Y Conversion

- Line currents for a Y-connected load:
  \[ I_a = \frac{V_{LN} \angle 0^\circ}{Z_Y} \]
  \[ I_b = \frac{V_{LN} \angle -120^\circ}{Z_Y} \]
  \[ I_c = \frac{V_{LN} \angle 120^\circ}{Z_Y} \]

- For a Δ-connected load:
  \[ I_a = \frac{3V_{LN} \angle 0^\circ}{Z_\Delta} \]
  \[ I_b = \frac{3V_{LN} \angle -120^\circ}{Z_\Delta} \]
  \[ I_c = \frac{3V_{LN} \angle 120^\circ}{Z_\Delta} \]
Δ – Y Conversion

- Equating any of the three line currents, we can determine the impedance relationship

\[
\frac{V_{LN} \angle 0^\circ}{Z_Y} = \frac{3V_{LN} \angle 0^\circ}{Z_\Delta}
\]

\[
Z_Y = \frac{Z_\Delta}{3}
\]

and

\[
Z_\Delta = 3Z_Y
\]
For balanced networks, we can simplify our analysis by considering only a single phase:

- A per-phase analysis
- Other phases are simply shifted by ±120°

For example, a balanced Y-Y circuit:
One-Line Diagrams

- Power systems are often depicted using one-line diagrams or single-line diagrams
  - Not a schematic – not all wiring is shown

- For example:
Example Problems
Determine:

- $I_a$
- $V_{AN}$
- $V_{AB}$
Determine:

- $I_a$
- $V_{AB}$
- $V_{AN}$
- $V_{BC}$
Power in Balanced $3\phi$ Networks
We’ll first determine the instantaneous power supplied by the source

- Neglecting line impedance, this is also the power absorbed by the load

The phase \( a \) line-to-neutral voltage is

\[
v_{an}(t) = \sqrt{2}V_{LN} \cos(\omega t + \delta)
\]

The phase \( a \) current is

\[
i_a(t) = \sqrt{2}I_L \cos(\omega t + \beta)
\]

where \( \beta \) depends on the load impedance
Instantaneous Power

- The instantaneous power delivered out of phase $a$ of the source is
  \[ p_a(t) = v_{an}(t)i_a(t) \]
  \[ p_a(t) = 2V_{LN}I_L \cos(\omega t + \delta) \cos(\omega t + \beta) \]
  \[ p_a(t) = V_{LN}I_L \cos(\delta - \beta) + V_{LN}I_L \cos(2\omega t + \delta + \beta) \]

- The $b$ and $c$ phases are shifted by $\pm 120^\circ$
  - Power from each of these phases is
    \[ p_b(t) = V_{LN}I_L \cos(\delta - \beta) + V_{LN}I_L \cos(2\omega t + \delta + \beta - 240^\circ) \]
    \[ p_c(t) = V_{LN}I_L \cos(\delta - \beta) + V_{LN}I_L \cos(2\omega t + \delta + \beta + 240^\circ) \]
The total power delivered by the source is the sum of the power from each phase

\[ p_{3\phi}(t) = p_a(t) + p_b(t) + p_c(t) \]

\[ p_{3\phi}(t) = 3V_{LN}I_L \cos(\delta - \beta) \]

\[ + V_{LN}I_L [\cos(2\omega t + \delta + \beta) \]
\[ + \cos(2\omega t + \delta + \beta - 240^\circ) \]
\[ + \cos(2\omega t + \delta + \beta + 240^\circ)] \]

Everything in the square brackets cancels, leaving

\[ p_{3\phi}(t) = 3V_{LN}I_L \cos(\delta - \beta) = P_{3\phi} \]

Power in a balanced 3φ network is constant

In terms of line-to-line voltages, the power is

\[ P_{3\phi} = \sqrt{3}V_{LL}I_L \cos(\delta - \beta) \]
The complex power delivered by phase $a$ is
\[ S_a = V_{an}I_a^* = V_{LN}∠\delta(I_L∠\beta)^* \]
\[ S_a = V_{LN}I_L∠(\delta - \beta) \]
\[ S_a = V_{LN}I_L \cos(\delta - \beta) + jV_{LN}I_L \sin(\delta - \beta) \]

For phase $b$, complex power is
\[ S_b = V_{bn}I_b^* = V_{LN}∠(\delta - 120°)(I_L∠(\beta - 120°))^* \]
\[ S_b = V_{LN}I_L∠(\delta - \beta) \]
\[ S_b = V_{LN}I_L \cos(\delta - \beta) + jV_{LN}I_L \sin(\delta - \beta) \]

This is equal to $S_a$ and also to phase $S_c$. 
The total complex power is

\[ S_{3\phi} = S_a + S_b + S_c \]

\[ S_{3\phi} = 3V_{LN} I_L \angle (\delta - \beta) \]

\[ S_{3\phi} = 3V_{LN} I_L \cos(\delta - \beta) + j3V_{LN} I_L \sin(\delta - \beta) \]

The apparent power is the magnitude of the complex power

\[ S_{3\phi} = 3V_{LN} I_L \]
Complex Power

- Complex power can be expressed in terms of the real and reactive power

\[ S_{3φ} = P_{3φ} + jQ_{3φ} \]

- The real power, as we’ve already seen is

\[ P_{3φ} = 3V_{LN}I_L \cos(δ − β) \]

- The reactive power is

\[ Q_{3φ} = 3V_{LN}I_L \sin(δ − β) \]
Advantages of Three-Phase Power

- Advantages of three-phase power:
  - For a given amount of power, *half the amount of wire required* compared to single-phase
    - No return current on neutral conductor
  - *Constant real power*
    - Constant motor torque
    - Less noise and vibration of machinery
Three-Phase Power – Example

- Determine
  - Load voltage, $V_{AB}$
  - Power triangle for the load
  - Power factor at the load

- We’ll do a per-phase analysis, so first convert the $\Delta$ load to a $Y$ load

$$Z_Y = \frac{Z_\Delta}{3} = 1 + j0.5\ \Omega$$
Three-Phase Power – Example

- The per-phase circuit:

  ![Per-phase circuit diagram]

- The line current is

  \[ I_L = \frac{V_{an}}{Z_L + Z_Y} = \frac{120 \angle 0^\circ V}{1.1 + j1 \Omega} = 80.7 \angle -42.3^\circ A \]

- The line-to-neutral voltage at the load is

  \[ V_{AN} = I_L Z_Y = (80.7 \angle -42.3^\circ A)(1 + j0.5 \Omega) \]

  \[ V_{AN} = (80.7 \angle -42.3^\circ A)(1.12 \angle 26.6^\circ \Omega) \]

  \[ V_{AN} = 90.25 \angle -15.71^\circ V \]
Three-Phase Power – Example

- Calculate the line-to-line voltage from the line-to-neutral voltage
  \[ V_{AB} = \sqrt{3} V_{AN} \angle 30^\circ \]
  \[ V_{AB} = 156 \angle 14.3^\circ V \]

- Alternatively, we could calculate line-to-line voltage from the two line-to-neutral voltages.
  - The line-to-neutral voltage at phase B is
    \[ V_{BN} = 90.25 \angle -135.71^\circ V \]
  - So the line-to-line voltage is given by
    \[ V_{AB} = V_{AN} - V_{BN} = 156 \angle 14.3^\circ V \]
Three-Phase Power – Example

- The complex power absorbed by the load is
  \[ S_{3\phi} = 3S_A = 3V_{AN}I_L^* \]
  \[ S_{3\phi} = 3(90.25\angle - 15.71^\circ V)(80.7\angle - 42.3^\circ A)^* \]
  \[ S_{3\phi} = 21.85 \angle 26.6^\circ kVA \]
  \[ S_{3\phi} = 19.53 + j9.78 kVA \]

- The apparent power:
  \[ S_{3\phi} = 21.85 kVA \]

- Real power:
  \[ P = 19.53 kW \]

- Reactive power:
  \[ Q = 9.78 kvar \]
Three-Phase Power – Example

- The power triangle at the load:

- The power factor at the load is

\[ p.f. = \cos(26.6^\circ) = \frac{P}{S} = \frac{19.53 \, kW}{21.85 \, kVA} \]

\[ p.f. = 0.89 \text{ lagging} \]
Example Problems
Determine complex power:
- From the source
- To the load
- To the line
Line-to-line voltage at the load is maintained at 4.16 kV.
What is the voltage at the source? How much complex power is delivered by the source?
Line-to-line voltage at the load is maintained at 4.16 kV.

Determine:
- Power factor at load
- Power triangle at load
- Loss in Lines
Add power factor correction to improve p.f. to 0.98, lagging.

Determine:

- Power triangle at load
- Loss in Lines