SECTION 8: FREQUENCY-RESPONSE DESIGN

ESE 430 – Feedback Control Systems
Introduction
Introduction

- In a previous section of notes, we saw how we can use root-locus techniques to design compensators.

- Two primary objectives of compensation:
  - Improve steady-state error
    - Proportional-integral (PI) compensation
    - Lag compensation
  - Improve dynamic response
    - Proportional-derivative (PD) compensation
    - Lead compensation

- In this section of notes, we’ll learn to design compensators using a system’s open-loop frequency response.
  - We’ll focus on lag and lead compensation.
Improving Steady-State Error
Consider the system above with a desired phase margin of $PM \approx 50^\circ$

According to the Bode plot:

- $\phi = -130^\circ$ at $\omega_{PM} = 3.46$ rad/sec
- Gain is $K_{PM} = -12.1$ dB at $\omega_{PM}$
- Set $K = -K_{PM} = 12.1$ dB = 4 for desired phase margin
Improving Steady-State Error

- Can read the position constant directly from the Bode plot: $K_p = 14.8 \text{ dB} \rightarrow 5.5$

- Note that $PM \approx 50^\circ$, as desired

- Gain margin is $GM = 17.9 \text{ dB}$
Improving Steady-State Error

- Steady-state error to a constant reference is

\[ e_{ss} = \frac{1}{1 + K_p} = 0.154 \rightarrow 15.4\% \]
Let’s say we want to reduce steady-state error to $e_{ss} < 5\%$

Required position constant

$$K_p > \frac{1}{0.05} - 1 = 19$$

Increase gain by 4x

- Bode plot shows desired position constant
- But, phase margin has been degraded significantly
Improving Steady-State Error

- Step response shows that error goal has been met
  - But, reduced phase margin results in significant overshoot and ringing
- Error improvement came at the cost of degraded phase margin
- Would like to be able to improve steady-state error without affecting phase margin
  - Integral compensation
  - Lag compensation

Closed-Loop Step Response

\( e_{ss} = 4.3\% \)
Integral Compensation
PI Compensation

- Proportional-integral (PI) compensator:
  \[ D(s) = \frac{1}{T_I} \frac{(T_I s + 1)}{s} \]

- Low-frequency gain increase
  - Infinite at DC
  - System type increase

- For \( \omega \gg 1/T_I \)
  - Gain unaffected
  - Phase affected little
  - PM unaffected

- Susceptible to integrator overflow
  - Lag compensation is often preferable
Lag Compensation
Lag Compensation

- Lag compensator
  \[ D(s) = \alpha \frac{(Ts + 1)}{(\alpha Ts + 1)} , \quad \alpha > 1 \]

- Objective: add a gain of \( \alpha \) at low frequencies without affecting phase margin

- Lower-frequency pole: \( s = -1/\alpha T \)

- Higher-frequency zero: \( s = -1/T \)

- Pole/zero spacing determined by \( \alpha \)

- For \( \omega \ll 1/\alpha T \)
  - Gain: \( \sim 20 \log(\alpha) \) dB
  - Phase: \( \sim 0^\circ \)

- For \( \omega \gg 1/T \)
  - Gain: \( \sim 0 \) dB
  - Phase: \( \sim 0^\circ \)
Lag Compensation vs. $\alpha$

- Gain increased at low frequency only
  - Dependent on $\alpha$
  - DC gain: $20\log(\alpha)$ dB
- Phase lag added between compensator pole and zero
  - $0^\circ \leq \phi_{\text{max}} \leq 90^\circ$
  - Dependent on $\alpha$
- Lag pole/zero well below crossover frequency
  - Phase margin unaffected

$$D(s) = \frac{\alpha (Ts + 1)}{(\alpha Ts + 1)}$$
Lag Compensator Design Procedure

- Lag compensator adds gain at low frequencies without affecting phase margin

- **Basic design procedure:**
  - Adjust gain to achieve the desired phase margin
  - Add compensation, increasing low-frequency gain to achieve desired error performance

- Same as adjusting gain to place poles at the desired damping on the root locus, then adding compensation
  - *Root locus is not changed*
  - Here, the *frequency response near the crossover frequency is not changed*
Lag Compensator Design Procedure

1. **Adjust gain**, $K$, of the *uncompensated* system to provide the *desired phase margin* plus 5° ... 10° (to account for small phase lag added by compensator)

2. Use the open-loop Bode plot for the uncompensated system with the value of gain set in the previous step to **determine the static error constant**

3. **Calculate** $\alpha$ as the low-frequency gain increase required to provide the desired error performance

4. **Set the upper corner frequency** (the zero) to be one decade below the crossover frequency: $1/T = \omega_{PM}/10$
   - Minimizes the added phase lag at the crossover frequency

5. **Calculate the lag pole**: $1/\alpha T$

6. **Simulate** and **iterate**, if necessary
Lag Example – Step 1

- Design a lag compensator for the above system to satisfy the following requirements
  - $e_{ss} < 2\%$ for a step input
  - $%OS \approx 12\%$

- First, determine the required phase margin to satisfy the overshoot requirement
  \[
  \zeta = - \frac{\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.559
  \]
  \[
  PM \approx 100\zeta = 55.9^\circ
  \]

- Add $\sim 10^\circ$ to account for compensator phase at $\omega_{PM}$
  \[
  PM = 65.9^\circ
  \]
Lag Example – Step 1

- Plot the open-loop Bode plot of the uncompensated system for $K = 1$
- Locate frequency where phase is $-180^\circ + PM = -114.1^\circ$
  - This is $\omega_{PM}$, the desired crossover frequency
  - $\omega_{PM} = 2.5 \text{ rad/sec}$
- Gain at $\omega_{PM}$ is $K_{PM}$
  - $K_{PM} = -8.4 \text{ dB} \rightarrow 0.38$
- Increase the gain by $1/K_{PM}$
  - $K = 8.4 \text{ dB} \rightarrow 2.63$
Gain has now been set to yield the desired phase margin of $PM = 65.9^\circ$.

Use the new open-loop bode plot to determine the static error constant.

Position constant of the uncompensated system given by the DC gain:

$$K_{pu} = 11.14 \text{ dB} \rightarrow 3.6$$
Lag Example – Step 3

- Calculate $\alpha$ to yield desired steady-state error improvement

- Steady-state error:

  \[ e_{ss} = \frac{1}{1 + K_p} < 0.02 \]

- The required position constant:

  \[ K_p > \frac{1}{e_{ss}} - 1 = 49 \rightarrow K_p = 50 \]

- Calculate $\alpha$ as the required position constant improvement

  \[ \alpha = \frac{K_p}{K_{pu}} = 13.9 \rightarrow \alpha = 14 \]
Lag Example – Steps 4 & 5

- Place the compensator zero one decade below the crossover frequency, $\omega_{PM} = 2.5 \text{ rad/sec}$
  
  \[ \frac{1}{T} = 0.25 \text{ rad/sec} \]
  \[ T = 4 \text{ sec} \]

- The compensator pole:
  
  \[ \frac{1}{\alpha T} = \frac{0.25}{14} \]
  \[ \frac{1}{\alpha T} = 0.018 \text{ rad/sec} \]

- Lag compensator transfer function
  
  \[ D(s) = \alpha \frac{(Ts + 1)}{(\alpha Ts + 1)} \]
  \[ D(s) = 14 \frac{(4s + 1)}{(56s + 1)} \]
Bode plot of compensated system shows:

- $PM = 60.5^\circ$
- $K_p = 50.5$
Lag Example – Step 6

- Lag compensator adds gain at low frequencies only
- Phase near the crossover frequency is nearly unchanged
Lag Example – Step 6

- Steady-state error requirement has been satisfied

- Overshoot spec has been met
  - Though slow tail makes overshoot assessment unclear
Lag Compensator – Summary

\[ D(s) = \alpha \frac{(Ts + 1)}{(\alpha Ts + 1)} \]

- **Higher-frequency zero:** \( s = -1/T \)
  - Place one decade below crossover frequency, \( \omega_{PM} \)

- **Lower-frequency pole:** \( s = -1/\alpha T \)
  - \( \alpha \) sets pole/zero spacing

- **DC gain:** \( \alpha \rightarrow 20 \log_{10}(\alpha) \) dB

- **Compensator adds low-frequency gain**
  - Static error constant improvement
  - Phase margin unchanged
Improving Dynamic Response
We’ve already seen two types of compensators to improve dynamic response:
- Proportional derivative (PD) compensation
- Lead compensation

Unlike with the lag compensator we just looked at, here, the objective is to alter the open-loop phase.

We’ll look briefly at PD compensation, but will focus on lead compensation.
Derivative Compensation
PD Compensation

- Proportional-Derivative (PD) compensator:
  \[ D(s) = (T_D s + 1) \]

- Phase added near (and above) the crossover frequency
  - Increased phase margin
  - Stabilizing effect

- Gain continues to rise at high frequencies
  - Sensor noise is amplified
  - Lead compensation is usually preferable
Lead Compensation
Lead Compensation

- With lead compensation, we have three design parameters:
  - **Crossover frequency**, $\omega_{PM}$
    - Determines closed-loop bandwidth, $\omega_{BW}$; risetime, $t_r$; peak time, $t_p$; and settling time, $t_s$
  - **Phase margin**, PM
    - Determines damping, $\zeta$, and overshoot
  - **Low-frequency gain**
    - Determines steady-state error performance

- We’ll look at the design of lead compensators for two common scenarios, either
  - Designing for **steady-state error** and **phase margin**, or
  - Designing for **bandwidth** and **phase margin**
Lead Compensation

- Lead compensator
  \[ D(s) = \frac{(Ts + 1)}{(\beta Ts + 1)} , \quad \beta < 1 \]

- Objectives: add phase lead near the crossover frequency and/or alter the crossover frequency

- Lower-frequency zero: \( s = -\frac{1}{T} \)

- Higher-frequency pole: \( s = -\frac{1}{\beta T} \)

- Zero/pole spacing determined by \( \beta \)

- For \( \omega \ll \frac{1}{T} \)
  - Gain: \( \sim 0 \) dB
  - Phase: \( \sim 0^\circ \)

- For \( \omega \gg \frac{1}{\beta T} \)
  - Gain: \( \sim 20 \log(1/\beta) \) dB
  - Phase: \( \sim 0^\circ \)
Lead Compensation vs. $\beta$

$$D(s) = \frac{(Ts + 1)}{(\beta Ts + 1)}$$, \quad \beta < 1

- $\beta$ determines:
  - Zero/pole spacing
  - Maximum compensator phase lead, $\phi_{max}$
  - High-frequency compensator gain
Lead Compensation – $\phi_{max}$

- $\beta$, zero/pole spacing, determines maximum phase lead

$$\phi_{max} = \sin^{-1} \left( \frac{1 - \beta}{1 + \beta} \right)$$

- Can use a desired $\phi_{max}$ to determine $\beta$

$$\beta = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})}$$

- $\phi_{max}$ occurs at $\omega_{max}$

$$\omega_{max} = \frac{1}{T \sqrt{\beta}}$$

$$T = \frac{1}{\omega_{max} \sqrt{\beta}}$$
Lead Compensation – Design Procedure

1. Determine loop gain, $K$, to satisfy either steady-state error requirements or bandwidth requirements:
   a) Set $K$ to provide the required static error constant, or
   b) Set $K$ to place the crossover frequency an octave below the desired closed-loop bandwidth

2. Evaluate the phase margin of the uncompensated system, using the value of $K$ just determined

3. If necessary, determine the required PM from $\zeta$ or overshoot specifications. Evaluate the PM of the uncompensated system and determine the required phase lead at the crossover frequency to achieve this PM. Add $\sim 10^\circ$ additional phase – this is $\phi_{max}$

4. Calculate $\beta$ from $\phi_{max}$

5. Set $\omega_{max} = \omega_{PM}$. Calculate $T$ from $\omega_{max}$ and $\beta$

6. Simulate and iterate, if necessary
Double-Lead Compensation

- A lead compensator can add, at most, 90° of phase lead.
- If more phase is required, use a double-lead compensator.

\[ D(s) = \left[ \frac{(Ts + 1)}{(\beta Ts + 1)} \right]^2 \]

- For phase lead over \( \sim 60^\circ \) ... \( 70^\circ \), \( 1/\beta \) must be very large, so typically use double-lead compensation.
Consider the following system

Design a compensator to satisfy the following
- $e_{ss} < 0.1$ for a ramp input
- $\%OS < 15\%$

Here, we’ll design a lead compensator to simultaneously adjust **low-frequency gain** and **phase margin**
Lead Example 1 – Steps 1 & 2

- The velocity constant for the uncompensated system is
  \[ K_v = \lim_{s \to 0} sK_G(s) \]
  \[ K_v = \lim_{s \to 0} \frac{K}{s + 1} = K \]
- Steady-state error is
  \[ e_{ss} = \frac{1}{K_v} < 0.1 \]
  \[ K_v = K > 10 \]
- Adding a bit of margin
  \[ K = 12 \]
- Bode plot shows the resulting phase margin is \( PM = 16.4^\circ \)
Lead Example 1 – Step 3

- Approximate required phase margin for $\%OS < 15$
  - Design for 13

- First calculate the required damping ratio

$$\zeta = -\frac{\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.545$$

- Approximate corresponding PM, and add $10^\circ$ correction factor

$$PM \approx 100\zeta + 10^\circ = 64.5^\circ$$

- Calculate the required phase lead

$$\phi_{max} = 64.5^\circ - 16.4^\circ = 48^\circ$$
Lead Example 1 – Steps 4 & 5

- Calculate $\beta$ from $\phi_{max}$

$$\beta = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})} = 0.147$$

- Set $\omega_{max} = \omega_{PM}$, as determined from Bode plot, and calculate $T$

$$\omega_{max} = \omega_{PM} = 3.4\text{ rad/sec}$$

$$T = \frac{1}{\omega_{max}\sqrt{\beta}} = \frac{1}{3.4\sqrt{0.169}} = 0.7687$$

- The resulting lead compensator transfer function is

$$KD(s) = K \frac{(Ts + 1)}{(\beta Ts + 1)} = 12 \frac{(0.7687s + 1)}{(0.1130s + 1)}$$
Lead Example 1 – Step 6

\[
D(s) = 12 \frac{(0.7687s + 1)}{(0.1130s + 1)}
\]

- The lead compensator Bode plot

![Lead Compensator Bode Plot](image)
Lead Example 1 – Step 6

- **Lead-compensated system:**
  - \( PM = 48.5^\circ \)
  - \( \omega_{PM} = 7.2 \text{ rad/sec} \)

- **High-frequency compensator gain increased the crossover frequency**
  - Phase was added at the *previous* crossover frequency
  - PM is below target

- **Move lead zero/pole to higher frequencies**
  - Reduce the crossover frequency increase
  - Improve phase margin
Lead Example 1 – Step 6

- As predicted by the insufficient phase margin, overshoot exceeds the target
  - \(\% OS = 20.9\% > 15\%\)

- Redesign compensator for higher \(\omega_{max}\)
  - Improve phase margin
  - Reduce overshoot
The steady-state error requirement has been satisfied

\[ e_{ss} = 0.08 < 0.1 \]

Will not change with compensator redesign

Low-frequency gain will not be changed
Lead Example 1 – Step 6

- Iteration yields acceptable value for $\omega_{max}$
  - $\omega_{max} = 5.5 \text{ rad/sec}$
  - Maintain same zero/pole spacing, $\beta$, and, therefore, same $\phi_{max}$

- Recalculate zero/pole time constants:
  
  $$T = \frac{1}{\omega_{max} \sqrt{\beta}} = \frac{1}{5.5 \sqrt{0.147}} = 0.4742$$

  $$\beta T = 0.147 \cdot 0.4742 = 0.0697$$

- The updated lead compensator transfer function:

  $$D(s) = 12 \frac{(0.4742s + 1)}{(0.0697s + 1)}$$
Crossover frequency has been reduced
\[ \omega_{PM} = 5.58 \text{ rad/sec} \]

Phase margin is close to the target
\[ PM = 58.2^\circ \]

Dip in phase is apparent, because \( \omega_{max} \) is now placed at point of lower open-loop phase
Lead Example 1 – Step 6

- Overshoot requirement now satisfied
  - $\%OS = 14.7\% < 15\%$

- Low-frequency gain has not been changed, so error requirement is still satisfied

- Design is complete
Lead Compensation – Example 2

- Again, consider the same system

- Design a compensator to satisfy the following
  - \( t_s \approx 1.2 \text{ sec} \ (\pm 1\%) \)
  - \( \%OS \approx 10\% \)

- Now, we’ll design a lead compensator to simultaneously adjust \textit{closed-loop bandwidth} and \textit{phase margin}
Lead Example 2 – Step 1

- The required damping ratio for 10% overshoot is
  \[ \zeta = -\frac{\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.5912 \]

- Given the required damping ratio, calculate the required closed-loop bandwidth to yield the desired settling time
  \[ \omega_{BW} = \frac{4.6}{t_s \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \]
  \[ \omega_{BW} = 7.52 \text{ rad/sec} \]

- We’ll initially set the gain, \( K \), to place the crossover frequency, \( \omega_{PM} \), one octave below the desired closed-loop bandwidth
  \[ \omega_{PM} = \omega_{BW}/2 = 3.8 \text{ rad/sec} \]
Lead Example 2 – Step 1

- Plot the Bode plot for $K = 1$
  - Determine the loop gain at the desired crossover frequency
    $$K_{PM} = -23.3 \text{ dB}$$
  - Adjust $K$ so that the loop gain at the desired crossover frequency is 0 dB
    $$K = \frac{1}{K_{PM}} = 23.3 \text{ dB} = 14.7$$
Lead Example 2 – Steps 2 & 3

- Generate a Bode plot using the gain value just determined
- Phase margin for the uncompensated system: 
  \[ PM_u = 14.9^\circ \]
- Required phase margin to satisfy overshoot requirement: 
  \[ PM \approx 100\zeta = 59.1^\circ \]
- Add 10° to account for crossover frequency increase 
  \[ PM = 69.1^\circ \]
- Required phase lead from the compensator 
  \[ \phi_{max} = PM - PM_u = 54.2^\circ \]
Lead Example 2 – Steps 4 & 5

- Calculate zero/pole spacing, $\beta$, from required phase lead, $\phi_{max}$
  
  $$\beta = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})} = 0.1040$$

- Calculate zero and pole time constants
  
  $$T = \frac{1}{\omega_{max}\sqrt{\beta}} = 0.8228 \text{ sec}$$
  
  $$\beta T = 0.0855 \text{ sec}$$

- The resulting lead compensator transfer function:
  
  $$KD(s) = K \frac{(Ts + 1)}{(\beta Ts + 1)}$$
  
  $$KD(s) = 14.7 \frac{(0.8228s + 1)}{(0.0855s + 1)}$$
Lead Example 2 – Step 6

- Bode plot of the compensated system
  - $PM = 49.9^\circ$
  - Substantially below target

- Crossover frequency is well above the desired value
  - $\omega_{PM} = 9.44 \text{ rad/sec}$

- Iteration will likely be required
Lead Example 2 – Step 6

- Overshoot exceeds the specified limit
  - $\%OS = 19.1\% > 10\%$

- Settling time is faster than required
  - $t_s = 0.98\ sec < 1.2\ sec$

- Iteration is required
  - Start by reducing the target $\omega_{PM}$
Lead Example 2 – Step 6

- Must redesign the compensator to meet specifications
  - Must increase PM to reduce overshoot
  - Can afford to reduce crossover, $\omega_{PM}$, to improve PM

- Try various combinations of the following
  - Reduce crossover frequency, $\omega_{PM}$
  - Increase compensator zero/pole frequencies, $\omega_{max}$
  - Increase added phase lead, $\phi_{max}$, by reducing $\beta$

- Iteration shows acceptable results for:
  - $\omega_{PM} = 2.4\, \text{rad/sec}$
  - $\omega_{max} = 3.4\, \text{rad/sec}$
  - $\phi_{max} = 52^\circ$
Lead Example 2 – Step 6

- Redesigned lead compensator:
  \[ KD(s) = 6.27 \frac{(0.8542s + 1)}{(0.1013s + 1)} \]

- Phase margin:
  \[ PM = 62^\circ \]

- Crossover frequency:
  \[ \omega_{PM} = 4.84 \text{ rad/sec} \]
Lead Example 2 – Step 6

- Dynamic response requirements are now satisfied
- Overshoot: 
  \[ \%OS = 8\% \]
- Settling time: 
  \[ t_s = 1.09 \text{ sec} \]
Lead Compensation – Example 2

- Lead compensator adds gain at higher frequencies
  - Increased crossover frequency
  - Faster response time

- Phase added near the crossover frequency
  - Improved phase margin
  - Reduced overshoot
Lead Compensation – Example 2

- Step response improvements:
  - Faster settling time
  - Faster risetime
  - Significantly less overshoot and ringing
Lead-Lag Compensation

- If performance specifications require adjustment of:
  - Bandwidth
  - Phase margin
  - Steady-state error

- Lead-lag compensation may be used

\[ KD(s) = \alpha \frac{(T_{lag}s + 1)}{(\alpha T_{lag}s + 1)} \frac{(T_{lead}s + 1)}{(\beta T_{lead}s + 1)} \]

- Many possible design procedures – one possibility:
  1. Design lag compensation to satisfy steady-state error and phase margin
  2. Add lead compensation to increase bandwidth, while maintaining phase margin