SECTION 2: THREE-PHASE POWER FUNDAMENTALS
AC Circuits & Phasors
AC Electrical Signals

- AC electrical signals (voltages and currents) are *sinusoidal*
  - Generated by rotating machinery
- Sinusoidal voltage (or current):
  \[ v(t) = V_p \cos(\omega t + \phi) \]  
  - This is a *time-domain* or *instantaneous* form expression
- Characterized by three parameters
  - Amplitude
  - Frequency
  - Phase
Amplitude

\[ v(t) = V_p \cos(\omega t + \phi) \]

- \( V_p \) in the above expression is **amplitude** or **peak voltage**
- We typically characterize power-system voltages and currents in terms of their **root-mean-square** (rms) values

\[
V_{rms} = \left( \frac{1}{T} \int_0^T v(t)^2 \, dt \right)^{\frac{1}{2}}
\]  

(2)

- A signal delivers the same power to a resistive load as a DC signal equal to its rms value
- For **sinusoids**:  

\[
V_{rms} = \frac{V_p}{\sqrt{2}}
\]  

(3)
Euler’s Identity

- Euler’s identity allows us to express sinusoidal signals as complex exponentials

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]  \hspace{1cm} (4)

so

\[ e^{j(\omega t+\phi)} = \cos(\omega t + \phi) + j \sin(\omega t + \phi) \]  \hspace{1cm} (5)

and

\[ V_p \cos(\omega t + \phi) = V_p Re\{e^{j(\omega t+\phi)}\} \]

\[ V_p \cos(\omega t + \phi) = \sqrt{2} V_{rms} Re\{e^{j(\omega t+\phi)}\} \]  \hspace{1cm} (6)
Phasor Representation

- **Phasor representation** simplifies circuit analysis when dealing with sinusoidal signals
  - Drop the time-harmonic (oscillatory) portion of the signal representation
    - Known and constant
  - Represent with *rms amplitude* and *phase* only

- For example, consider the time-domain voltage expression
  \[ v(t) = \sqrt{2} \, V_{rms} \cos(\omega t + \phi) \]

- The phasor representation, in *exponential* form, is
  \[ V = V_{rms} e^{j\phi} \]

- Can also express in *polar* or *Cartesian* form
  \[ V = V_{rms} \angle \phi = V_{rms} \cos(\phi) + jV_{rms} \sin(\phi) \]

- In these notes **bold** type will be used to distinguish phasors
- We’ll always assume rms values for phasor magnitudes
Phasors

- Think of a phasor as a vector in the complex plane
  - Has **magnitude** and **angle**

- Circuit analysis in the phasor domain is simplified
  - Derivative and integrals become algebraic expressions

- Consider the voltage across inductance and capacitance:

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<th>Time Domain</th>
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<td>Capacitor</td>
<td>( v(t) = \frac{1}{c} \int i(t)dt )</td>
<td>( V = \frac{1}{j\omega C} I )</td>
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<tr>
<td>Inductor</td>
<td>( v(t) = L \frac{di}{dt} )</td>
<td>( V = j\omega LI )</td>
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<tr>
<td>Resistor</td>
<td>( v(t) = i(t)R )</td>
<td>( V = IR )</td>
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Phasors

- In general, in the phasor domain
  \[ V = IZ \] (7)

  and
  \[ I = \frac{V}{Z} \]

- Ohm’s law

- \( Z \) is a complex impedance
  - Not a phasor, but also expressed in exponential, polar, or Cartesian form
**Phasors - Example**

- Determine $i(t)$ and $v_L(t)$ for the following circuit, driven by a $120 \, V_{rms}$, $60 \, Hz$ source.

- At $60 \, Hz$ the inductor impedance is
  \[ jX_L = j\omega L = j2\pi \cdot 60 \, Hz \cdot 5 \, mH = j1.88 \, \Omega \]

- The total impedance seen by the source is
  \[ Z = R + jX_L = 2 + j1.88 \, \Omega \]

- Converting to polar form
  \[ Z = |Z| \angle \theta \]
  \[ |Z| = \sqrt{R^2 + X^2} = 2.74 \, \Omega \]
  \[ \theta = \tan^{-1}\left(\frac{X}{R}\right) = 43^\circ \]
  \[ Z = 2.74 \angle 43^\circ \, \Omega \]
Phasors – Example

- The source voltage is
  \[ v(t) = \sqrt{2} \cdot 120V \cos(2\pi \cdot 60Hz \cdot t) \]

- The source voltage phasor is
  \[ V = 120 \angle 0^\circ V \]

- The current phasor is
  \[ I = \frac{V}{Z} = \frac{120 \angle 0^\circ V}{2.74 \angle 43^\circ \Omega} = 43.7 \angle -43^\circ A \]

- We can use the current phasor to determine the phasor for the voltage across the resistor
  \[ V_L = IR = (43.7 \angle -43^\circ) \cdot 2\Omega \]
  \[ V_L = 87.4 \angle -43^\circ V \]
We have phasor representations for desired quantities

\[ I = 43.7 \angle -43^\circ \text{ A} \]

\[ V_L = 87.4 \angle -43^\circ \text{ V} \]

We can now convert these to their time-domain expressions

\[ i(t) = \sqrt{2} \cdot 43.7 \text{ A} \cdot \cos(2\pi \cdot 60\text{Hz} \cdot t - 43^\circ) \]

\[ v(t) = \sqrt{2} \cdot 87.4 \text{ V} \cdot \cos(2\pi \cdot 60\text{Hz} \cdot t - 43^\circ) \]
Phasor Diagrams
Phasor Diagrams

- Phasors are complex values
  - Magnitude and phase
  - Vectors in the complex plane
  - Can represent graphically

- Phasor diagram
  - Graphical representation of phasors in a circuit
  - KVL and Ohm’s law expressed graphically
Phasor Diagram – Example 1

- **Source voltage is the reference phasor**
  \( V_S = 120 \angle 0^\circ \, V \)

- Its phasor diagram:

- **Ohm’s law gives the current**
  \[
  I = \frac{V_S}{2 + j2 \Omega} = 42.2 \angle -45^\circ \, A
  \]

- Adding to the phasor diagram:
Phasor Diagram – Example 1

- Ohm’s law gives the inductor voltage
  \[ V_L = I \cdot j \omega L = (42.2 \angle -45^\circ \text{A}) \cdot j2 \Omega \]
  \[ V_L = 85 \angle 45^\circ \text{V} \]

- Finally, KVL gives \( V_R \)
  \[ V_R = V_S - V_L \]
  \[ V_R = 120 \angle 0^\circ \text{V} - 85 \angle 45^\circ \text{V} \]
  \[ V_R = 85 \angle -45^\circ \]
Phasor Diagram – Example 2

- Source voltage is the reference phasor
  \[ V_S = 2.4\angle0^\circ \, kV \]

- Ohm’s law gives the current
  \[ I = \frac{V_S}{3.5 + j3 \, \Omega} = 521\angle-41^\circ \, A \]
Phasor Diagram – Example 2

- **Ohm’s law gives the resistor voltage**
  \[ V_{\text{Line}_R} = I \cdot R \]
  \[ V_{\text{Line}_R} = (521 \angle -41^\circ \text{ A}) \cdot 1.5 \ \Omega \]
  \[ V_{\text{Line}_R} = 781 \angle -41^\circ \ \text{V} \]

- **KVL gives** \( V_2 \)
  \[ V_2 = V_S - V_{\text{Line}_R} \]
  \[ V_2 = 2.4 \angle 0^\circ \text{ kV} - 781 \angle -41^\circ \ \text{V} \]
  \[ V_2 = 1.88 \angle 15.7^\circ \text{ kV} \]
**Phasor Diagram – Example 2**

- **Drop across the inductor:**
  \[ V_{LineL} = (521\angle -41^\circ \text{ A}) \cdot j2 \, \Omega \]
  \[ V_{LineL} = 1.04\angle 49^\circ \text{ kV} \]

- **KVL gives the voltage across the load**
  \[ V_R = V_2 - V_{LineL} \]
  \[ V_R = 1.88\angle 15.7^\circ \text{ kV} - 1.04\angle 49^\circ \text{ kV} \]
  \[ V_R = 1.16\angle -14^\circ \text{ kV} \]
Phasor Diagram – Example 2

- Alternatively, treat the line as a single impedance

\[ V_{\text{Line}} = I \cdot Z_{\text{Line}} \]
\[ V_{\text{Line}} = (521\angle -41^\circ \ A) \cdot (1.5 + j2 \ \Omega) \]
\[ V_{\text{LineL}} = 1.3\angle 12.5^\circ \ kV \]

- KVL gives the voltage across the load

\[ V_R = V_S - V_{\text{Line}} \]
\[ V_R = 2.4\angle 0^\circ kV - 1.3\angle 12.5^\circ kV \]
\[ V_R = 1.16\angle -14^\circ \ kV \]
Power – Real Power & Power Factor
The overall goal of a power distribution network is to transfer power from a source to loads.

**Instantaneous power:**

- Power supplied by a source or absorbed by a load or network element as a function of time

\[ p(t) = v(t) \cdot i(t) \]  

The nature of this instantaneous power flow is determined by the impedance of the load.

Next, we’ll look at the instantaneous power delivered to loads of different impedances.
The voltage across the resistive load is
\[ v(t) = V_p \cos(\omega t + \delta) \]

Current through the resistor is
\[ i(t) = \frac{V_p}{R} \cos(\omega t + \delta) \]

The instantaneous power absorbed by the resistor is
\[ p_R(t) = v(t) \cdot i(t) = V_p \cos(\omega t + \delta) \cdot \frac{V_p}{R} \cos(\omega t + \delta) \]
\[ p_R(t) = \frac{V_p^2}{R} \cos^2(\omega t + \delta) = \frac{V_p^2}{R} \left[ 1 + \cos(2\omega t + 2\delta) \right] \]
Instantaneous Power – Resistive Load

\[ p_R(t) = \frac{V_p^2}{2R} [1 + \cos(2\omega t + 2\delta)] \]

- Making use of the rms voltage

\[ p_R(t) = \frac{(\sqrt{2} V_{\text{rms}})^2}{2R} [1 + \cos(2\omega t + 2\delta)] \]

\[ p_R(t) = \frac{V_{\text{rms}}^2}{R} [1 + \cos(2\omega t + 2\delta)] \quad (9) \]

- The instantaneous power absorbed by the resistor has a non-zero average value and a double-frequency component
Instantaneous Power – Resistive Load

- Power delivered to the resistive load has a non-zero average value and a double-frequency component.
Now consider the power absorbed by a purely capacitive load

Again, \( v(t) = V_p \cos(\omega t + \delta) \)

The current flowing to the load is

\[
i(t) = I_p \cos(\omega t + \delta + 90^\circ)
\]

where

\[
I_p = \frac{V_p}{X_C} = \frac{V_p}{1/\omega C} = \omega CV_p
\]

The instantaneous power delivered to the capacitive load is

\[
p_C(t) = v(t) \cdot i(t)
\]

\[
p_C(t) = V_p \cos(\omega t + \delta) \cdot \omega CV_p \cos(\omega t + \delta + 90^\circ)
\]
Instantaneous Power – Capacitive Load

\[ p_C(t) = \omega CV_p^2 \frac{1}{2} \left[ \cos(-90^\circ) + \cos(2\omega t + 2\delta + 90^\circ) \right] \]

\[ p_C(t) = \omega C \frac{V_p^2}{2} \cdot \cos(2\omega t + 2\delta + 90^\circ) \]

- In terms of rms voltage
  \[ p_C(t) = \omega CV_{rms}^2 \cdot \cos(2\omega t + 2\delta + 90^\circ) \]

- This is a double frequency sinusoid, but, unlike for the resistive load, the average value is zero
Now consider the power absorbed by a purely inductive load.

Now the load current *lags* by $90^\circ$:

$$i(t) = I_p \cos(\omega t + \delta - 90^\circ)$$

where

$$I_p = \frac{V_p}{X_L} = \frac{V_p}{\omega L}$$

The instantaneous power delivered to the inductive load is

$$p_L(t) = v(t) \cdot i(t)$$

$$p_L(t) = V_p \cos(\omega t + \delta) \cdot \frac{V_p}{\omega L} \cos(\omega t + \delta - 90^\circ)$$
Instantaneous Power – Inductive Load

\[ p_L(t) = \frac{V_p^2}{\omega L} \left[ \cos(90°) + \cos(2\omega t + 2\delta - 90°) \right] \]

\[ p_L(t) = \frac{V_p^2}{2\omega L} \cdot \cos(2\omega t + 2\delta - 90°) \]

- In terms of rms voltage
  \[ p_L(t) = \frac{V_{rms}^2}{\omega L} \cdot \cos(2\omega t + 2\delta - 90°) \]

- As for the capacitive load, this is a double frequency sinusoid with an average value of zero
Finally, consider the instantaneous power absorbed by a general RLC load.

Phase angle of the current is determined by the angle of the impedance:

\[ i(t) = I_p \cos(\omega t + \beta) \]

The instantaneous power is

\[ p(t) = V_p \cos(\omega t + \delta) \cdot I_p \cos(\omega t + \beta) \]

\[ p(t) = \frac{V_p I_p}{2} \left[ \cos(\delta - \beta) + \cos(2\omega t + \delta + \beta) \right] \]

\[ p(t) = V_{rms} I_{rms} \left[ \cos(\delta - \beta) + \cos(2\omega t + 2\delta - (\delta - \beta)) \right] \]
Using the following trig identity

\[ \cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B) \]

we get

\[
p(t) = V_{rms}I_{rms}[\cos(\delta - \beta) + \cos(\delta - \beta) \cos(2\omega t + 2\delta) \\
+ \sin(\delta - \beta) \sin(2\omega t + 2\delta)]
\]

and

\[
p(t) = V_{rms}I_{rms} \cos(\delta - \beta) [1 + \cos(2\omega t + 2\delta)] \\
+ V_{rms}I_{rms} \sin(\delta - \beta) \sin(2\omega t + 2\delta)
\]
Letting

\[ I_R = I_{rms} \cos(\delta - \beta) \quad \text{and} \quad I_X = I_{rms} \sin(\delta - \beta) \]

we have

\[ p(t) = V_{rms}I_R[1 + \cos(2\omega t + 2\delta)] + V_{rms}I_X \sin(2\omega t + 2\delta) \]  \hspace{1cm} (12)

There are two components to the power:

\[ p_R(t) = V_{rms}I_R[1 + \cos(2\omega t + 2\delta)] \]  \hspace{1cm} (13)

is the power absorbed by the resistive component of the load, and

\[ p_X(t) = V_{rms}I_X \sin(2\omega t + 2\delta) \]  \hspace{1cm} (14)

is the power absorbed by the reactive component of the load.
Real Power

According to (9) and (13), power delivered to a resistance has a non-zero average value.

- Purely resistive load or a load with a resistive component

This is real power, average power, or active power:

\[ P = V_{rms}I_R \]

\[ P = V_{rms}I_{rms} \cos(\delta - \beta) \]  \hspace{1cm} (15)

- Real power has units of watts (W)
- Real power is power that results in work (or heat dissipation)
Power Factor

- The phase angle \( (\delta - \beta) \) represents the phase difference between the voltage and the current
  - This is the **power factor angle**
  - The angle of the load impedance

- Note that the *real power* is a function of the *cosine of the power factor angle*

\[
P = V_{rms}I_{rms} \cos(\delta - \beta)
\]

- This is the **power factor**

\[
p.f. = \cos(\delta - \beta) \quad (16)
\]

- For a purely resistive load, voltage and current are in phase

\[
p.f. = \cos(\delta - \beta) = \cos(0^\circ) = 1
\]
\[
P = V_{rms}I_{rms}
\]
Power Factor

- For a purely capacitive load, current leads the voltage by 90°
  \[ p.f. = \cos(\delta - \beta) = \cos(-90°) = 0 \]
  \[ P = 0 \]
  - This is referred to as a \textit{leading power factor}
  - Power factor is \textit{leading} for loads with \textit{capacitive} reactance

- For a purely inductive load, current lags the voltage by 90°
  \[ p.f. = \cos(\delta - \beta) = \cos(90°) = 0 \]
  \[ P = 0 \]
  - Loads with inductive reactance have \textit{lagging} power factors

- Note that power factor is defined to always be \textit{positive}
  \[ 0 \leq p.f. \leq 1 \]
Reactive & Complex Power
Reactive Power

- The other part of instantaneous power, as given by (12), is the power delivered to the reactive component of the load

\[ p_x(t) = V_{\text{rms}} I_{\text{rms}} \sin(\delta - \beta) \sin(2\omega t + 2\delta) \]

- Unlike real power, this component of power has zero average value

- The amplitude is the **reactive power**

\[ Q = V_{\text{rms}} I_{\text{rms}} \sin(\delta - \beta) \text{ var} \]

- Units are **volts-amperes reactive**, or **var**

- Power that flows to and from the load reactance
  - Does not result in work or heat dissipation
Complex Power

- **Complex power** is defined as the product of the rms voltage phasor and conjugate rms current phasor

\[
S = VI^*
\]  

(18)

where the voltage has phase angle \(\delta\)

\[
V = V_{rms} \angle \delta
\]

and the current has phase angle \(\beta\)

\[
I = I_{rms} \angle \beta \quad \rightarrow \quad I^* = I_{rms} \angle -\beta
\]

- The complex power is

\[
S = VI^* = (V_{rms} \angle \delta)(I_{rms} \angle -\beta)
\]

\[
S = V_{rms}I_{rms} \angle (\delta - \beta)
\]

(19)
Complex Power

- Complex power has units of \textit{volts-amperes} (VA)
- The \textit{magnitude} of complex power is \textit{apparent power}
  \[ S = V_{\text{rms}}I_{\text{rms}} \text{ VA} \] (20)
- Apparent power also has units of volts-amperes
- Complex power is the vector sum of real power (in phase with $V$) and reactive power ($\pm 90^\circ$ out of phase with $V$)
  \[ S = P + jQ \] (21)
Complex Power

- **Real power** can be expressed in terms of complex power
  
  \[ P = \text{Re}\{S\} \]

  or in terms of **apparent power**
  
  \[ P = S \cdot \cos(\delta - \beta) = S \cdot \text{p. f.} \]

- Similarly, **reactive power**, is the imaginary part of complex power
  
  \[ Q = \text{Im}\{S\} \]

  and can also be related to **apparent power**
  
  \[ Q = S \cdot \sin(\delta - \beta) \]

- And, **power factor** is the **ratio** between **real power** and **apparent power**
  
  \[ \text{p. f.} = \cos(\delta - \beta) = \frac{P}{S} \]
Passive Sign Convention
Applying a consistent sign convention allows us to easily determine whether network elements supply or absorb real and reactive power.

- **Passive sign convention** or **load convention**
  - Positive current defined to enter the positive voltage terminal of an element

- If $P > 0$ or $Q > 0$, then real or reactive power is *absorbed* by the element.

- If $P < 0$ or $Q < 0$, then real or reactive power is *supplied* by the element.
Power Absorbed by Passive Elements

- Complex power absorbed by a **resistor**

\[ S_R = VI_R^* = (V \angle \delta) \left( \frac{V}{R} \angle -\delta \right) \]
\[ S_R = \frac{V^2}{R} \]

- Positive and purely real
  - Resistors *absorb real* power
  - Reactive power is zero

- Complex power absorbed by a **capacitor**

\[ S_C = VI_C^* = (V \angle \delta) (-j\omega CV \angle -\delta) \]
\[ S_C = -j\omega CV^2 \]

- Negative and purely imaginary
  - Capacitors *supply reactive* power
  - Real power is zero
Power Absorbed by Passive Elements

- Complex power absorbed by an **inductor**

\[
S_L = VI_L^* = (V \angle \delta) \left( \frac{V}{-j\omega L} \angle -\delta \right)
\]

\[
S_L = j \frac{V^2}{\omega L}
\]

- Positive and purely imaginary
  - Inductors *absorb reactive* power
  - *Real* power is *zero*

- In summary:
  - Resistors absorb real power, zero reactive power
  - Capacitors supply reactive power, zero real power
  - Inductors absorb reactive power, zero real power
Power Triangle
Power Triangle

- Complex power is the vector sum of real power (in phase with $V$) and reactive power ($\pm 90^\circ$ out of phase with $V$)

\[
S = P + jQ
\]

- Complex, real, and reactive powers can be represented graphically, as a power triangle

\[
P = VI \cos(\delta - \beta) \text{ W}
\]

\[
Q = VI \sin(\delta - \beta) \text{ var}
\]

\[
S = VI^* \text{ VA}
\]
Quickly and graphically provides power information

- Power factor and power factor angle
- Leading or lagging power factor
- Reactive nature of the load – capacitive or inductive
Lagging Power Factor

- For loads with inductive reactance
  - Impedance angle is positive
  - Power factor angle is positive
  - Power factor is lagging

\[
\begin{align*}
S &= VI^* \ \text{VA} \\
Q &= VI \sin(\delta - \beta) \ \text{var} \\
P &= VI \cos(\delta - \beta) \ \text{W}
\end{align*}
\]

- \(Q\) is positive
  - The load absorbs reactive power
Leading Power Factor

- For loads with **capacitive** reactance
  - Impedance angle is negative
  - Power factor angle is negative
  - Power factor is **leading**

\[
P = VI \cos(\delta - \beta) \text{ W} \\
Q = VI \sin(\delta - \beta) \text{ var}
\]

- \( Q \) is negative
  - The load **supplies** reactive power
Power Factor Correction
The overall goal of power distribution is to supply power to do work

- Real power

Reactive power does not perform work, but

- Must be supplied by the source
- Still flows over the lines

For a given amount of real power consumed by a load, we’d like to

- Reduce reactive power, $Q$
- Reduce $S$ relative to $P$, that is
- Reduce the p.f. angle, and
- Increase the p.f.

*Power factor correction*
Power Factor Correction – Example

- Consider a source driving an inductive load
- Determine:
  - Real power absorbed by the load
  - Reactive power absorbed by the load
  - p.f. angle and p.f.
- Draw the power triangle

- Current through the resistance is
  \[ I_R = \frac{120 \, V}{3 \, \Omega} = 40 \, A \]

- Current through the inductance is
  \[ I_L = \frac{120 \, V}{j2 \, \Omega} = 60 \angle -90^\circ \, A \]

- The total load current is
  \[ I = I_R + I_L = (40 - j60) \, A = 72.1 \angle -56.3^\circ \, A \]
The power factor angle is

\[ \theta = (\delta - \beta) = 0° - (-56.3°) \]

\[ \theta = 56.3° \]

The power factor is

\[ p.f. = \cos(\theta) = \cos(56.3°) \]

\[ p.f. = 0.55 \text{ lagging} \]

Real power absorbed by the load is

\[ P = VI \cos(\theta) = 120 \, V \cdot 72.1 \, A \cdot 0.55 \]

\[ P = 4.8 \, kW \]

Alternatively, recognizing that real power is power absorbed by the resistance

\[ P = VI_R = 120 \, V \cdot 40 \, A = 4.8 \, kW \]
Power Factor Correction – Example

- Reactive power absorbed by the load is
  \[ Q = VI \sin(\theta) = 120 \, V \cdot 72.1 \, A \cdot 0.832 \]
  \[ Q = 7.2 \, kvar \]

- This is also the power absorbed by the load inductance
  \[ Q = VI_L = 120 \, V \cdot 60 \, A = 7.2 \, kvar \]

- Apparent power is
  \[ S = VI = 120 \, V \cdot 72.1 \, A = 8.65 \, kVA \]

- Or, alternatively
  \[ S = \sqrt{P^2 + Q^2} \]
  \[ S = \sqrt{(4.8 \, kW)^2 + (7.2 \, kvar)^2} = 8.65 \, kVA \]
Power Factor Correction – Example

☐ The *power triangle*:

☐ Here, the source is supplying 4.8 kW at a power factor of 0.55 lagging

☐ Let’s say we want to reduce the apparent power supplied by the source

☐ Deliver 4.8 kW at a p.f. of 0.9 lagging

☐ Add *power factor correction*

☐ Add capacitors to *supply* reactive power
For $p.f. = 0.9$, we need a power factor angle of

$$\theta' = \cos^{-1}(0.9) = 25.8^\circ$$

Power factor correction will help flatten the power triangle:
Power Factor Correction – Example

- Reactive power to the power-factor-corrected load is reduced from $Q$ to $Q'$
  
  \[ Q' = P \tan(\theta') \]
  
  \[ Q' = 4.8 \text{ kW} \cdot \tan(25.8^\circ) \]
  
  \[ Q' = 2.32 \text{ kvar} \]

- The required reactive power absorbed (negative, so it is supplied) by the capacitors is

  \[ Q_c = Q' - Q = 2.32 \text{ kvar} - 7.2 \text{ kvar} \]
  
  \[ Q_c = -4.88 \text{ kvar} \]
Power Factor Correction – Example

- Reactive power absorbed by the capacitor is
  \[ Q_C = \frac{V^2}{X_C} \]

- So the required capacitive reactance is
  \[ X_C = \frac{V^2}{Q_C} = \frac{(120 \, V)^2}{-4.88 \, \text{kvar}} = -2.95 \, \Omega \]

- The addition of \(-j2.95 \, \Omega\) provides the desired power factor correction
The source voltage in the circuit is

\[ v(t) = \sqrt{2} \cdot 120V \cos(2\pi \cdot 60Hz \cdot t). \]

Determine the complex power delivered to the load.
Two three-phase load are connected in parallel:
- 50 kVA at a power factor of 0.9, leading
- 125 kW at a power factor of 0.85, lagging.

Draw the power triangle and determine the combined power factor.
Power is delivered to a single-phase load with an impedance of $Z_L = 3 + j2 \, \Omega$ at 120 V. Add power factor correction in parallel with the load to yield a power factor of 0.95, lagging.

Determine the reactive power and impedance of the power factor correction component.
Draw a phasor diagram for the following circuit.

- Draw a phasor for the voltage across each component and for the current.
- Apply KVL graphically. That is, add the individual component phasors together graphically to show that the result is equal to the source voltage phasor.
Balanced Three-Phase Networks
We are accustomed to *single-phase* power in our homes and offices

A single *line* voltage referenced to a *neutral*

Electrical power is generated, transmitted, and largely consumed (by industrial customers) as *three-phase power*

Three individual line voltages and (possibly) a neutral

Line voltages all differ in phase by ±120°
Δ- and Y-Connected Networks

- Two possible three-phase configurations
  - Applies to both sources and loads

Y-Connected Source

Δ-Connected Source

- Y-connected network has a neutral node
- Δ-connected network has no neutral
Line-to-Neutral Voltages

- In the Y network, voltages $V_{an}$, $V_{bn}$, and $V_{cn}$ are **line-to-neutral voltages**
- A three-phase source is **balanced** if
  - Line-to-neutral voltages have equal magnitudes
  - Line-to-neutral voltage are each $120^\circ$ out of phase with one another
- A three-phase network is balanced if
  - Sources are balanced
  - The impedances connected to each phase are equal
The line-to-neutral voltages are

\[ V_{an} = V_{LN} \angle 0^\circ \]
\[ V_{bn} = V_{LN} \angle -120^\circ \]
\[ V_{cn} = V_{LN} \angle -240^\circ = V_{LN} \angle +120^\circ \]

This is a *positive-sequence* or *abc-sequence* source

- \( V_{an} \) leads \( V_{bn} \), which leads \( V_{cn} \)

Can also have a *negative- or acb-sequence* source

- \( V_{an} \) leads \( V_{cn} \), which leads \( V_{bn} \)

We’ll always assume *positive-sequence* sources
The voltages between the three phases are **line-to-line voltages**

Apply KVL to relate line-to-line voltages to line-to-neutral voltages

\[ V_{ab} - V_{an} + V_{bn} = 0 \]
\[ V_{ab} = V_{an} - V_{bn} \]

We know that

\[ V_{an} = V_{LN} \angle 0° \]
and

\[ V_{bn} = V_{LN} \angle -120° \]

so

\[ V_{ab} = V_{LN} \angle 0° - V_{LN} \angle -120° = V_{LN} (1 \angle 0° - 1 \angle -120°) \]
\[ V_{ab} = V_{LN} \left[ 1 - \left( -\frac{1}{2} - j \frac{\sqrt{3}}{2} \right) \right] = V_{LN} \left[ \frac{3}{2} + j \frac{\sqrt{3}}{2} \right] \]

\[ V_{ab} = \sqrt{3}V_{LN} \angle 30° \]
Again applying KVL, we can find $V_{bc}$

$$V_{bc} = V_{bn} - V_{cn}$$

$$V_{bc} = V_{LN} \angle -120^\circ - V_{LN} \angle 120^\circ$$

$$V_{bc} = V_{LN} \left[ \left( -\frac{1}{2} - j \frac{\sqrt{3}}{2} \right) - \left( -\frac{1}{2} + j \frac{\sqrt{3}}{2} \right) \right]$$

$$V_{bc} = V_{LN} (-j \sqrt{3})$$

$$V_{bc} = \sqrt{3}V_{LN} \angle -90^\circ$$

Similarly,

$$V_{ca} = \sqrt{3}V_{LN} \angle 150^\circ$$
Line-to-Line Voltages

- The line-to-line voltages, with $V_{an}$ as the reference:
  
  $$V_{ab} = \sqrt{3}V_{LN} \angle 30^\circ$$
  $$V_{bc} = \sqrt{3}V_{LN} \angle -90^\circ$$
  $$V_{ca} = \sqrt{3}V_{LN} \angle 150^\circ$$

- Line-to-line voltages are $\sqrt{3}$ times the line-to-neutral voltage

- Can also express in terms of individual line-to-neutral voltages:
  
  $$V_{ab} = \sqrt{3}V_{an} \angle 30^\circ$$
  $$V_{bc} = \sqrt{3}V_{bn} \angle 30^\circ$$
  $$V_{ca} = \sqrt{3}V_{cn} \angle 30^\circ$$
Currents in Three-Phase Networks
Line Currents in Balanced $3\phi$ Networks

- We can use the line-to-neutral voltages to determine the line currents
  - Y-connected source and load
  - Balanced load – all impedances are equal: $Z_Y$

$\begin{align*}
I_a &= \frac{V_{AN}}{Z_Y} = \frac{V_{LN} \angle 0^\circ}{Z_Y} \\
I_b &= \frac{V_{BN}}{Z_Y} = \frac{V_{LN} \angle -120^\circ}{Z_Y} \\
I_c &= \frac{V_{CN}}{Z_Y} = \frac{V_{LN} \angle +120^\circ}{Z_Y}
\end{align*}$

- Line currents are balanced as long as the source and load are balanced
Neutral Current in Balanced $3\phi$ Networks

- Apply KCL to determine the neutral current

$$I_n = I_a + I_b + I_c$$

$$I_n = \frac{V_{LN}}{Z_Y} [1\angle0^\circ + 1\angle-120^\circ + 1\angle120^\circ]$$

$$I_n = \frac{V_{LN}}{Z_Y} \left[ 1 + \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \right]$$

$$I_n = 0$$

- The neutral conductor carries no current in a balanced three-phase network
Y- and Δ-connected Loads
Three-Phase Load Configurations

- As for sources, three-phase loads can also be connected in two different configurations.

**Y-Connected Load**

- The Y load has a neutral connection, but the Δ load does not.

**Δ-Connected Load**

- Currents in a Y-connected load are the line currents we just determined.
- Next, we’ll look at currents in a Δ-connected load.
Balanced Δ-Connected Loads

We can use line-to-line voltages to determine the currents in Δ-connected loads.

\[ I_{AB} = \frac{V_{AB}}{Z_\Delta} = \frac{\sqrt{3}V_{AN} \angle 30^\circ}{Z_\Delta} = \frac{\sqrt{3}V_{LN} \angle 30^\circ}{Z_\Delta} \]

\[ I_{BC} = \frac{V_{BC}}{Z_\Delta} = \frac{\sqrt{3}V_{BN} \angle 30^\circ}{Z_\Delta} = \frac{\sqrt{3}V_{LN} \angle -90^\circ}{Z_\Delta} \]

\[ I_{CA} = \frac{V_{CA}}{Z_\Delta} = \frac{\sqrt{3}V_{CN} \angle 30^\circ}{Z_\Delta} = \frac{\sqrt{3}V_{LN} \angle 150^\circ}{Z_\Delta} \]
Balanced $\Delta$-Connected Loads

- Applying KCL, we can determine the line currents

\[ I_a = I_{AB} - I_{CA} \]

\[ I_a = \frac{\sqrt{3}V_{LN}}{Z_\Delta} [1\angle30^\circ - 1\angle150^\circ] \]

\[ I_a = \frac{\sqrt{3}V_{LN}}{Z_\Delta} \left[ \left( \frac{\sqrt{3}}{2} + j\frac{1}{2} \right) - \left( -\frac{\sqrt{3}}{2} + j\frac{1}{2} \right) \right] = \frac{\sqrt{3}V_{LN}}{Z_\Delta} [\sqrt{3}] = \frac{3V_{LN}}{Z_\Delta} \]

- The other line currents can be found similarly:

\[
\begin{align*}
I_a &= \frac{3V_{LN}\angle0^\circ}{Z_\Delta} = \sqrt{3}I_{AB}\angle-30^\circ \\
I_b &= \frac{3V_{LN}\angle-120^\circ}{Z_\Delta} = \sqrt{3}I_{BC}\angle-30^\circ \\
I_c &= \frac{3V_{LN}\angle120^\circ}{Z_\Delta} = \sqrt{3}I_{CA}\angle-30^\circ
\end{align*}
\]
Δ-Y Conversion
Δ – Y Conversion

- Analysis is often simpler when dealing with Y-connected loads
  - Would like a way to convert Δ loads to Y loads (and vice versa)

- For a Y load and a Δ load to be equivalent, they must result in equal line currents
\[ \Delta - Y \text{ Conversion} \]

- **Line currents for a \( Y \)-connected load:**
  
  \[
  I_a = \frac{V_{LN} \angle 0^\circ}{Z_Y} \\
  I_b = \frac{V_{LN} \angle -120^\circ}{Z_Y} \\
  I_c = \frac{V_{LN} \angle 120^\circ}{Z_Y}
  \]

- **For a \( \Delta \)-connected load:**
  
  \[
  I_a = \frac{3V_{LN} \angle 0^\circ}{Z_\Delta} \\
  I_b = \frac{3V_{LN} \angle -120^\circ}{Z_\Delta} \\
  I_c = \frac{3V_{LN} \angle 120^\circ}{Z_\Delta}
  \]
Δ – Y Conversion

- Equating any of the three line currents, we can determine the impedance relationship

\[
\begin{align*}
\frac{V_{LN} \angle 0^\circ}{Z_Y} &= \frac{3V_{LN} \angle 0^\circ}{Z_\Delta} \\
Z_Y &= \frac{Z_\Delta}{3} \quad \text{and} \quad Z_\Delta = 3Z_Y
\end{align*}
\]
Per-Phase Analysis
For balanced networks, we can simplify our analysis by considering only a single phase.

- A per-phase analysis
- Other phases are simply shifted by ±120°

For example, a balanced $Y$-$Y$ circuit:
One-Line Diagrams

- Power systems are often depicted using **one-line diagrams** or **single-line diagrams**
  - Not a schematic – not all wiring is shown

- For example:
Example Problems
Given the following balanced 3-\( \phi \) quantities:

\[ V_{BC} = 480 \angle 15^\circ \text{ and } I_B = 21 \angle -28^\circ \]

Find:

1) \( V_{AB} \)
2) \( V_{AN} \)
3) \( I_A \)
4) \( I_C \)
Find:

- Per-phase circuit
- Line current, $I_A$
- Load voltage
Find:

- Per-phase circuit
- Line current, $I_A$
- L-L and L-N load voltages
Power in Balanced 3φ Networks
We’ll first determine the instantaneous power supplied by the source

- Neglecting line impedance, this is also the power absorbed by the load

- The phase \( a \) line-to-neutral voltage is
  \[
  v_{an}(t) = \sqrt{2}V_{LN} \cos(\omega t + \delta)
  \]

- The phase \( a \) current is
  \[
  i_a(t) = \sqrt{2}I_L \cos(\omega t + \beta)
  \]
  where \( \beta \) depends on the load impedance
The instantaneous power delivered out of phase $\alpha$ of the source is

$$p_a(t) = v_{an}(t)i_a(t)$$
$$p_a(t) = 2V_{LN}I_L \cos(\omega t + \delta) \cos(\omega t + \beta)$$
$$p_a(t) = V_{LN}I_L \cos(\delta - \beta) + V_{LN}I_L \cos(2\omega t + \delta + \beta)$$

The $b$ and $c$ phases are shifted by $\pm 120^\circ$.

Power from each of these phases is

$$p_b(t) = V_{LN}I_L \cos(\delta - \beta) + V_{LN}I_L \cos(2\omega t + \delta + \beta - 240^\circ)$$
$$p_c(t) = V_{LN}I_L \cos(\delta - \beta) + V_{LN}I_L \cos(2\omega t + \delta + \beta + 240^\circ)$$
Instantaneous Power

- The total power delivered by the source is the sum of the power from each phase

\[ p_{3\phi}(t) = p_a(t) + p_b(t) + p_c(t) \]

\[ p_{3\phi}(t) = 3V_{LN}I_L \cos(\delta - \beta) + V_{LN}I_L[\cos(2\omega t + \delta + \beta) + \cos(2\omega t + \delta + \beta - 240^\circ) + \cos(2\omega t + \delta + \beta + 240^\circ)] \]

- Everything in the square brackets cancels, leaving

\[ p_{3\phi}(t) = 3V_{LN}I_L \cos(\delta - \beta) = P_{3\phi} \]

- Power in a balanced 3\( \phi \) network is constant

- In terms of line-to-line voltages, the power is

\[ P_{3\phi} = \sqrt{3}V_{LL}I_L \cos(\delta - \beta) \]
Complex Power

- The **complex power** delivered by phase $a$ is

  $$S_a = V_{an}I_a^* = V_{LN} \angle \delta (I_L \angle \beta)^*$$
  $$S_a = V_{LN}I_L \angle (\delta - \beta)$$
  $$S_a = V_{LN}I_L \cos(\delta - \beta) + jV_{LN}I_L \sin(\delta - \beta)$$

- For phase $b$, complex power is

  $$S_b = V_{bn}I_b^* = V_{LN} \angle (\delta - 120^\circ) (I_L \angle (\beta - 120^\circ))^*$$
  $$S_b = V_{LN}I_L \angle (\delta - \beta)$$
  $$S_b = V_{LN}I_L \cos(\delta - \beta) + jV_{LN}I_L \sin(\delta - \beta)$$

- This is equal to $S_a$ and also to phase $S_c$
The total complex power is

\[ S_{3\phi} = S_a + S_b + S_c \]

\[ S_{3\phi} = 3V_{LN}I_L \angle (\delta - \beta) \]

\[ S_{3\phi} = 3V_{LN}I_L \cos(\delta - \beta) + j3V_{LN}I_L \sin(\delta - \beta) \]

The apparent power is the magnitude of the complex power

\[ S_{3\phi} = 3V_{LN}I_L \]
Complex Power

- Complex power can be expressed in terms of the real and reactive power
  \[ S_{3\Phi} = P_{3\Phi} + jQ_{3\Phi} \]
  
- The **real power**, as we’ve already seen is
  \[ P_{3\Phi} = 3V_{LN}I_L \cos(\delta - \beta) \]
  
- The **reactive power** is
  \[ Q_{3\Phi} = 3V_{LN}I_L \sin(\delta - \beta) \]
Advantages of Three-Phase Power

- Advantages of three-phase power:
  - For a given amount of power, *half the amount of wire required* compared to single-phase
    - No return current on neutral conductor
  - *Constant real power*
    - Constant motor torque
    - Less noise and vibration of machinery
Three-Phase Power – Example

- Determine
  - Load voltage, $V_{AB}$
  - Power triangle for the load
  - Power factor at the load

- We’ll do a per-phase analysis, so first convert the Δ load to a Y load

$$Z_Y = \frac{Z_\Delta}{3} = 1 + j0.5 \, \Omega$$
The per-phase circuit:

The line current is

\[
I_L = \frac{V_{an}}{Z_L + Z_Y} = \frac{120\angle0^\circ V}{1.1 + j1 \Omega} = \frac{120\angle0^\circ V}{1.45\angle42.3^\circ \Omega}
\]

\[
I_L = 80.7\angle -42.3^\circ A
\]

The line-to-neutral voltage at the load is

\[
V_{AN} = I_L Z_Y = (80.7\angle -42.3^\circ A)(1 + j0.5 \Omega)
\]

\[
V_{AN} = (80.7\angle -42.3^\circ A)(1.12\angle26.6^\circ \Omega)
\]

\[
V_{AN} = 90.25\angle -15.71^\circ V
\]
Three-Phase Power – Example

- The line-to-line load voltage is
  \[ V_{AB} = \sqrt{3} V_{AN} \angle 30^\circ \]
  \[ V_{AB} = 156 \angle 14.3^\circ \text{ V} \]

- Alternatively, we could calculate line-to-line voltage from phase \( A \) and phase \( B \) line-to-neutral voltages
  \[ V_{AB} = V_{AN} - V_{BN} \]
  \[ V_{AB} = 90.25 \angle -15.71^\circ \text{ V} - 90.25 \angle -135.71^\circ \text{ V} \]
  \[ V_{AB} = 156 \angle 14.3^\circ \text{ V} \]
Three-Phase Power – Example

- The complex power absorbed by the load is
  \[ S_{3\phi} = 3S_A = 3V_{AN}I_L^* \]
  \[ S_{3\phi} = 3(90.25 \angle - 15.71^\circ \ V)(80.7 \angle - 42.3^\circ \ A)^* \]
  \[ S_{3\phi} = 21.85 \angle 26.6^\circ \ kVA \]
  \[ S_{3\phi} = 19.53 + j9.78 \ kVA \]

- The apparent power:
  \[ S_{3\phi} = 21.85 \ kVA \]

- Real power:
  \[ P = 19.53 \ kW \]

- Reactive power:
  \[ Q = 9.78 \ kvar \]
Three-Phase Power – Example

- The power triangle at the load:

- The power factor at the load is

\[ p.f. = \cos(26.6^\circ) = \frac{P}{S} = \frac{19.53 \text{ kW}}{21.85 \text{ kVA}} \]

\[ p.f. = 0.89 \text{ lagging} \]
Example Problems
Find:
- Source power
- Source power factor
- Load power
- Load power factor
Find:
- Source power
- Load power
- Power lost in lines