Introduction

- We have seen how to design feedback control systems using the *root locus*.
- In this section of the course, we’ll learn how to do the same using the open-loop *frequency response*.

**Objectives:**
- Determine static error constants from the open-loop frequency response.
- Determine closed-loop stability from the open-loop frequency response.
- Use the open-loop frequency response for compensator design to:
  - Improve steady-state error.
  - Improve transient response.
Steady-State Error from Bode Plots
For unity-feedback systems, open-loop transfer function gives \textit{static error constants}.

Use static error constants to calculate \textit{steady-state error}:

\begin{align*}
K_p &= \lim_{s \to 0} G(s) \\
K_v &= \lim_{s \to 0} sG(s) \\
K_a &= \lim_{s \to 0} s^2G(s)
\end{align*}

We can also determine static error constants from a system’s \textit{open-loop Bode plot}.
Static Error Constant – Type 0

- For a type 0 system
  \[ K_p = \lim_{{s \to 0}} G(s) \]

- At low frequency, i.e. below any open-loop poles or zeros
  \[ G(s) \approx K_p \]

- Read \( K_p \) directly from the open-loop Bode plot
  - Low-frequency gain

\[ G(s) = \frac{100(s + 30)}{(s + 3)(s + 200)} \]

\[ K_p = 14.0 \text{ dB} \rightarrow 5.0 \]
Static Error Constant – Type 1

- For a type 1 system
  \[ K_v = \lim_{s \to 0} sG(s) \]

- At low frequencies, i.e. below any other open-loop poles or zeros
  \[ G(s) \approx \frac{K_v}{s} \quad \text{and} \quad |G(j\omega)| \approx \frac{K_v}{\omega} \]

- A straight line with a slope of \(-20 \text{ dB/dec}\)

- Evaluating this low-frequency asymptote at \(\omega = 1\) yields the velocity constant, \(K_v\)

- On the Bode plot, extend the low-frequency asymptote to \(\omega = 1\)
  - Gain of this line at \(\omega = 1\) is \(K_v\)
Static Error Constant – Type 1

\[ G(s) = \frac{85(s + 0.1)(s + 50)}{s(s^2 + 10s + 125)} \]

\[ K_v = 10.6 \text{ dB} \rightarrow 3.4 \]
Static Error Constant – Type 2

- For a type 2 system
  \[ K_a = \lim_{s \to 0} s^2 G(s) \]

- At low frequencies, i.e. below any other open-loop poles or zeros
  \[ G(s) \approx \frac{K_a}{s^2} \quad \text{and} \quad |G(j\omega)| \approx \frac{K_a}{\omega^2} \]

- A straight line with a slope of \(-40\) dB/dec

- Evaluating this low-frequency asymptote at \(\omega = 1\) yields the acceleration constant, \(K_a\)

- On the Bode plot, extend the low-frequency asymptote to \(\omega = 1\)
  - Gain of this line at \(\omega = 1\) is \(K_a\)
Static Error Constant – Type 2

\[ G(s) = \frac{1600(s + 0.1)(s + 5)}{s^2(s + 100)} \]

\[ K_a = 18.1 \text{ dB} \rightarrow 8.0 \]
Stability from Open-Loop Bode Plots
Consider the following system

We already have a couple of tools for assessing stability as a function of loop gain, $K$

- Routh Hurwitz
- Root locus

Root locus:
- Stable for some values of $K$
- Unstable for others
Stability

- In this case gain is stable **below** some value
- Other systems may be stable for gain **above** some value
- Marginal stability point:
  - Closed-loop poles on the imaginary axis at $\pm j\omega_1$
  - For gain $K = K_1$
Open-Loop Frequency Response & Stability

- Marginal stability point occurs when closed-loop poles are on the imaginary axis
  - Angle criterion satisfied at $\pm j\omega_1$
    $$|KG(j\omega_1)| = 1 \quad \text{and} \quad \angle KG(j\omega_1) = -180^\circ$$
  - Note that $-180^\circ = 180^\circ$

- $KG(j\omega)$ is the open-loop frequency response

- Marginal stability occurs when:
  - Open-loop gain is: $|KG(j\omega)| = 0 \text{ dB}$
  - Open-loop phase is: $\angle KG(j\omega) = -180^\circ$
Stability from Open-Loop Bode Plots

- Varying $K$ simply shifts gain response up or down
- Here, stable for smaller gain values
  - $|KG(j\omega)| < 0 \text{ dB}$ when $\angle KG(j\omega) = -180^\circ$
- Often, stable for larger gain values
  - $|KG(j\omega)| > 0 \text{ dB}$ when $\angle KG(j\omega) = -180^\circ$
- Root locus provides this information
  - Bode plot does not
Open-Loop Frequency Response & Stability

- **Open-loop Bode plot** can be used to assess stability
  - But, we need to know if system is closed-loop stable for low gain or high gain

- Here, we’ll assume **open-loop-stable systems**
  - Closed-loop stable for low gain

- Open-loop Bode plot can tell us:
  - Is a system closed-loop stable?
  - If so, how stable?
    - I.e. how close to marginal stability

- **Two stability metrics:**
  - Gain margin
  - Phase margin
Stability Margins
Crossover Frequencies

- Two important frequencies when assessing stability:
  - **Gain crossover frequency**, $\omega_{PM}$
    - The frequency at which the open-loop gain crosses $0 \, dB$
  - **Phase crossover frequency**, $\omega_{GM}$
    - The frequency at which the open-loop phase crosses $-180^\circ$
Gain Margin

- An open-loop-stable system will be closed-loop stable as long as its gain is less than unity at the phase crossover frequency.

- **Gain margin, GM**
  - The change in open-loop gain at the phase crossover frequency required to make the closed-loop system unstable.
Phase Margin

- An open-loop-stable system will be closed-loop stable as long as its phase has not fallen below $-180^\circ$ at the gain crossover frequency.

- **Phase margin, PM**
  - The change in open-loop phase at the gain crossover frequency required to make the closed-loop system unstable.
Gain and Phase Margins from Bode Plots

GM and PM from Bode Plots

Gain [dB]

Phase [deg]

Frequency [rad/sec]
PM can be expressed as a function of damping ratio, $\zeta$, as

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}ight)$$

For $PM \leq 65^\circ$ or so, we can approximate:

$$PM \approx 100\zeta \quad \text{or} \quad \zeta \approx \frac{PM}{100}$$
Frequency Response Analysis in MATLAB
```matlab
[mag, phase] = bode(sys, w)
```

- **sys**: system model – state-space, transfer function, or other
- **w**: optional frequency vector – in rad/sec
- **mag**: system gain response vector
- **phase**: system phase response vector – in degrees

- If no outputs are specified, bode response is automatically plotted – preferable to plot yourself
- Frequency vector input is optional
  - If not specified, MATLAB will generate automatically

- May need to do: `squeeze(mag)` and `squeeze(phase)` to eliminate singleton dimensions of output matrices
[GM, PM, wgm, wpm] = margin(sys)

- sys: system model – state-space, transfer function, or other
- GM: gain margin
- PM: phase margin – in degrees
- wgm: frequency at which GM is measured, the phase crossover frequency – in rad/sec
- wpm: frequency at which PM is measured, the gain crossover frequency

- If no outputs are specified, a Bode plot with GM and PM indicated is automatically generated
Frequency-Response Design
Frequency-Response Design

- In a previous section of notes, we saw how we can use root-locus techniques to design compensators.

- Two primary objectives of compensation:
  - Improve steady-state error:
    - Proportional-integral (PI) compensation
    - Lag compensation
  - Improve dynamic response:
    - Proportional-derivative (PD) compensation
    - Lead compensation

- Now, we’ll learn to design compensators using a system’s *open-loop frequency response*. We’ll focus on lag and lead compensation.
Improving Steady-State Error
Consider the system above with a desired phase margin of $PM \approx 50^\circ$

According to the Bode plot:

- $\phi = -130^\circ$ at $\omega_{PM} = 3.46 \, \text{rad/sec}$
- Gain is $K_{PM} = -12.1 \, \text{dB}$ at $\omega_{PM}$
- Set $K = -K_{PM} = 12.1 \, \text{dB} = 4$ for desired phase margin
Improving Steady-State Error

- Can read the position constant directly from the Bode plot: $K_p = 14.8 \text{ dB} \rightarrow 5.5$

- Note that $PM \approx 50^\circ$, as desired

- Gain margin is $GM = 17.9 \text{ dB}$
Improving Steady-State Error

- Steady-state error to a constant reference is

$$e_{ss} = \frac{1}{1 + K_p} = 0.154 \rightarrow 15.4\%$$
Let’s say we want to reduce steady-state error to \( e_{ss} < 5\% \)

Required position constant

\[ K_p > \frac{1}{0.05} - 1 = 19 \]

Increase gain by 4x

Bode plot shows desired position constant

But, phase margin has been degraded significantly
Improving Steady-State Error

- Step response shows that error goal has been met
  - But, reduced phase margin results in significant overshoot and ringing

- Error improvement came at the cost of degraded phase margin

- Would like to be able to improve steady-state error without affecting phase margin
  - Integral compensation
  - Lag compensation
Integral Compensation
PI Compensation

- Proportional-integral (PI) compensator:
  \[ D(s) = \frac{1}{T_D} \left( \frac{T_D s + 1}{s} \right) \]

- Low-frequency gain increase
  - Infinite at DC
  - System type increase

- For \( \omega \gg 1/T_D \)
  - Gain unaffected
  - Phase affected little
  - PM unaffected

- Susceptible to integrator overflow
  - Lag compensation is often preferable
Lag Compensation
Lag Compensation

- Lag compensator
  \[ D(s) = \alpha \frac{(Ts + 1)}{(\alpha Ts + 1)} , \quad \alpha > 1 \]

- Objective: add a gain of \( \alpha \) at low frequencies without affecting phase margin

- Lower-frequency pole: \( s = -1/\alpha T \)
- Higher-frequency zero: \( s = -1/T \)
- Pole/zero spacing determined by \( \alpha \)

- For \( \omega \ll 1/\alpha T \)
  - Gain: \( \sim 20 \log(\alpha) \) dB
  - Phase: \( \sim 0^\circ \)

- For \( \omega \gg 1/T \)
  - Gain: \( \sim 0 \) dB
  - Phase: \( \sim 0^\circ \)
Lag Compensation vs. $\alpha$

- Gain increased at low frequency only
  - Dependent on $\alpha$
  - DC gain: $20\log(\alpha)$ dB

- Phase lag added between compensator pole and zero
  - $0^\circ \leq \phi_{max} \leq 90^\circ$
  - Dependent on $\alpha$

- Lag pole/zero well below crossover frequency
  - Phase margin unaffected

\[ D(s) = \alpha \frac{(Ts + 1)}{(\alpha Ts + 1)} \]
Lag Compensator Design Procedure

- Lag compensator adds gain at low frequencies without affecting phase margin

- **Basic design procedure:**
  - Adjust gain to achieve the desired phase margin
  - Add compensation, increasing low-frequency gain to achieve desired error performance

- Same as adjusting gain to place poles at the desired damping on the root locus, then adding compensation
  - *Root locus is not changed*
  - Here, the *frequency response near the crossover frequency is not changed*
Lag Compensator Design Procedure

1. Adjust gain, $K$, of the uncompensated system to provide the desired phase margin plus 5° ... 10° (to account for small phase lag added by compensator)

2. Use the open-loop Bode plot for the uncompensated system with the value of gain set in the previous step to determine the static error constant

3. Calculate $\alpha$ as the low-frequency gain increase required to provide the desired error performance

4. Set the upper corner frequency (the zero) to be one decade below the crossover frequency: $1/T = \omega_{PM}/10$
   - Minimizes the added phase lag at the crossover frequency

5. Calculate the lag pole: $1/\alpha T$

6. Simulate and iterate, if necessary
Lag Example – Step 1

- Design a lag compensator for the above system to satisfy the following requirements
  - $e_{ss} < 2\%$ for a step input
  - $\%OS \approx 12\%$

- First, determine the required phase margin to satisfy the overshoot requirement
  
  \[
  \zeta = -\frac{\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.559
  \]

  \[
  PM \approx 100\zeta = 55.9^\circ
  \]

- Add $\sim 10^\circ$ to account for compensator phase at $\omega_{PM}$
  
  \[
  PM = 65.9^\circ
  \]
Lag Example – Step 1

- Plot the open-loop Bode plot of the uncompensated system for $K = 1$
- Locate frequency where phase is $-180^\circ + PM = -114.1^\circ$
  - This is $\omega_{PM}$, the desired crossover frequency
  - $\omega_{PM} = 2.5 \text{ rad/sec}$
- Gain at $\omega_{PM}$ is $K_{PM}$
  - $K_{PM} = -8.4 \text{ dB} \rightarrow 0.38$
- Increase the gain by $1/K_{PM}$
  - $K = 8.4 \text{ dB} \rightarrow 2.63$
Gain has now been set to yield the desired phase margin of $PM = 65.9^\circ$

Use the new open-loop bode plot to determine the static error constant

Position constant of the uncompensated system given by the DC gain:

$$K_{pu} = 11.14 \text{ dB} \rightarrow 3.6$$
Lag Example – Step 3

- Calculate $\alpha$ to yield desired steady-state error improvement

  Steady-state error:
  
  $$e_{ss} = \frac{1}{1 + K_p} < 0.02$$

- The required position constant:

  $$K_p > \frac{1}{e_{ss}} - 1 = 49 \rightarrow K_p = 50$$

- Calculate $\alpha$ as the required position constant improvement

  $$\alpha = \frac{K_p}{K_{pu}} = 13.9 \rightarrow \alpha = 14$$
Lag Example – Steps 4 & 5

- Place the compensator zero one decade below the crossover frequency, $\omega_{PM} = 2.5 \text{ rad/sec}$

  \[ \frac{1}{T} = 0.25 \text{ rad/sec} \]
  \[ T = 4 \text{ sec} \]

- The compensator pole:

  \[ \frac{1}{\alpha T} = \frac{0.25}{14} \]
  \[ \frac{1}{\alpha T} = 0.018 \text{ rad/sec} \]

- Lag compensator transfer function

  \[ D(s) = \alpha \frac{(Ts + 1)}{\alpha Ts + 1} \]
  \[ D(s) = 14 \frac{(4s + 1)}{(56s + 1)} \]
Lag Example – Step 6

- Bode plot of compensated system shows:
  - $PM = 60.5^\circ$
  - $K_p = 50.5$
Lag Example – Step 6

- Lag compensator adds gain at low frequencies only
- Phase near the crossover frequency is nearly unchanged
Lag Example – Step 6

☐ Steady-state error requirement has been satisfied

☐ Overshoot spec has been met

☐ Though slow tail makes overshoot assessment unclear
Lag Compensator – Summary

\[ D(s) = \alpha \frac{(Ts + 1)}{(\alpha Ts + 1)} \]

- Higher-frequency zero: \( s = -1/T \)
  - Place one decade below crossover frequency, \( \omega_{PM} \)

- Lower-frequency pole: \( s = -1/\alpha T \)
  - \( \alpha \) sets pole/zero spacing

- DC gain: \( \alpha \rightarrow 20 \log_{10}(\alpha) \) dB

- Compensator adds low-frequency gain
  - Static error constant improvement
  - Phase margin unchanged
Improving Dynamic Response
Improving Dynamic Response

- We’ve already seen two types of compensators to improve dynamic response:
  - Proportional derivative (PD) compensation
  - Lead compensation

- Unlike with the lag compensator we just looked at, here, the objective is to alter the open-loop phase.

- We’ll look briefly at PD compensation, but will focus on lead compensation.
Derivative Compensation
PD Compensation

- Proportional-Derivative (PD) compensator:

\[ D(s) = (T_D s + 1) \]

- Phase added near (and above) the crossover frequency
  - Increased phase margin
  - Stabilizing effect

- Gain continues to rise at high frequencies
  - Sensor noise is amplified
  - Lead compensation is usually preferable
Lead Compensation
Lead Compensation

- With lead compensation, we have three design parameters:
  - **Crossover frequency**, $\omega_{PM}$
    - Determines closed-loop bandwidth, $\omega_{BW}$; risetime, $t_r$; peak time, $t_p$; and settling time, $t_s$
  - **Phase margin**, PM
    - Determines damping, $\zeta$, and overshoot
  - **Low-frequency gain**
    - Determines steady-state error performance

- We’ll look at the design of lead compensators for two common scenarios, *either*
  - Designing for *steady-state error* and *phase margin*, or
  - Designing for *closed-loop bandwidth* and *phase margin*
Lead Compensation

- **Lead compensator**
  \[ D(s) = \frac{(Ts + 1)}{(\beta Ts + 1)} , \quad \beta < 1 \]

- **Objectives**: add phase lead near the crossover frequency and/or alter the crossover frequency

- **Lower-frequency zero**: \( s = -\frac{1}{T} \)

- **Higher-frequency pole**: \( s = -\frac{1}{\beta T} \)

- **Zero/pole spacing determined by** \( \beta \)

- **For** \( \omega \ll \frac{1}{T} \)
  - Gain: \(~0 \text{ dB}\)
  - Phase: \(~0°\)

- **For** \( \omega \gg \frac{1}{\beta T} \)
  - Gain: \(~20 \log(1/\beta) \text{ dB}\)
  - Phase: \(~0°\)
Lead Compensation vs. $\beta$

$$D(s) = \frac{(Ts + 1)}{(\beta Ts + 1)}$$, \quad \beta < 1

- $\beta$ determines:
  - Zero/pole spacing
  - Maximum compensator phase lead, $\phi_{max}$
  - High-frequency compensator gain
Lead Compensation – $\phi_{max}$

- $\beta$, zero/pole spacing, determines maximum phase lead

$$\phi_{max} = \sin^{-1}\left(\frac{1 - \beta}{1 + \beta}\right)$$

- Can use a desired $\phi_{max}$ to determine $\beta$

$$\beta = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})}$$

- $\phi_{max}$ occurs at $\omega_{max}$

$$\omega_{max} = \frac{1}{T \sqrt{\beta}}$$

$$T = \frac{1}{\omega_{max} \sqrt{\beta}}$$
Lead Compensation – Design Procedure

1. Determine loop gain, $K$, to satisfy either steady-state error requirements or bandwidth requirements:
   a) Set $K$ to provide the required static error constant, or
   b) Set $K$ to place the crossover frequency an octave below the desired closed-loop bandwidth

2. Evaluate the phase margin of the uncompensated system, using the value of $K$ just determined

3. If necessary, determine the required PM from $\zeta$ or overshoot specifications. Evaluate the PM of the uncompensated system and determine the required phase lead at the crossover frequency to achieve this PM. Add $\sim 10^\circ$ additional phase – this is $\phi_{max}$

4. Calculate $\beta$ from $\phi_{max}$

5. Set $\omega_{max} = \omega_{PM}$. Calculate $T$ from $\omega_{max}$ and $\beta$

6. Simulate and iterate, if necessary
**Closed-Loop Bandwidth and Transient Response**

- **Closed-loop bandwidth**, $\omega_{BW}$, is one possible design criterion
  - How is it related to transient response?

- For a **second-order system** (or approximate second-order system):
  - Closed-loop bandwidth and **damping ratio** and **natural frequency**, $\zeta$ and $\omega_n$

\[
\omega_{BW} = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}
\]

- Closed-loop bandwidth and $\pm 1\%$ **settling time**, $t_s$

\[
\omega_{BW} \approx \frac{4.6}{t_s \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}
\]

- Closed-loop bandwidth and **peak time**, $t_p$

\[
\omega_{BW} = \frac{4}{t_p \sqrt{1 - \zeta^2}} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}
\]
Double-Lead Compensation

- A lead compensator can add, at most, 90° of phase lead.
- If more phase is required, use a double-lead compensator.

\[ D(s) = \left[ \frac{(Ts + 1)}{(\beta Ts + 1)} \right]^2 \]

- For phase lead over \(\sim 60^\circ \ldots 70^\circ\), \(1/\beta\) must be very large, so typically use double-lead compensation.
Lead Compensation – Example 1

- Consider the following system

- Design a compensator to satisfy the following
  - $e_{ss} < 0.1$ for a ramp input
  - $\%OS < 15\%$

- Here, we’ll design a lead compensator to simultaneously adjust low-frequency gain and phase margin
The velocity constant for the uncompensated system is

\[
K_v = \lim_{s \to 0} sK_G(s)
\]

\[
K_v = \lim_{s \to 0} \frac{K}{s + 1} = K
\]

Steady-state error is

\[
e_{ss} = \frac{1}{K_v} < 0.1
\]

\[
K_v = K > 10
\]

Adding a bit of margin

\[
K = 12
\]

Bode plot shows the resulting phase margin is \( PM = 16.4^\circ \)
Lead Example 1 – Step 3

- Approximate required phase margin for $\%OS < 15$
  - Design for $13$

- First calculate the required damping ratio

$$\zeta = -\frac{\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.545$$

- Approximate corresponding PM, and add $10^\circ$ correction factor

$$PM \approx 100\zeta + 10^\circ = 64.5^\circ$$

- Calculate the required phase lead

$$\phi_{max} = 64.5^\circ - 16.4^\circ = 48^\circ$$
Lead Example 1 – Steps 4 & 5

- Calculate $\beta$ from $\phi_{max}$

$$\beta = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})} = 0.147$$

- Set $\omega_{max} = \omega_{PM}$, as determined from Bode plot, and calculate $T$

$$\omega_{max} = \omega_{PM} = 3.4 \text{ rad/sec}$$

$$T = \frac{1}{\omega_{max} \sqrt{\beta}} = \frac{1}{3.4 \sqrt{0.147}} = 0.766$$

- The resulting lead compensator transfer function is

$$KD(s) = K \frac{(Ts + 1)}{(\beta Ts + 1)} = 12 \frac{(0.766s + 1)}{(0.113s + 1)}$$
Lead Example 1 – Step 6

\[ KD(s) = 12 \frac{(0.766s + 1)}{(0.113s + 1)} \]

- The lead compensator Bode plot
Lead Example 1 – Step 6

- Lead-compensated system:
  - $PM = 48.5^\circ$
  - $\omega_{PM} = 7.2 \text{ rad/sec}$

- High-frequency compensator gain increased the crossover frequency
  - Phase was added at the previous crossover frequency
  - PM is below target

- Move lead zero/pole to higher frequencies
  - Reduce the crossover frequency increase
  - Improve phase margin
Lead Example 1 – Step 6

- As predicted by the insufficient phase margin, overshoot exceeds the target
  - \(\%OS = 20.9\% > 15\%\)

- Redesign compensator for higher \(\omega_{max}\)
  - Improve phase margin
  - Reduce overshoot
Lead Example 1 – Step 6

- The steady-state error requirement has been satisfied
  - $e_{ss} = 0.08 < 0.1$

- Will not change with compensator redesign
  - Low-frequency gain will not be changed
Lead Example 1 – Step 6

- Iteration yields acceptable value for $\omega_{max}$
  - $\omega_{max} = 5.5$ rad/sec
  - Maintain same zero/pole spacing, $\beta$, and, therefore, same $\phi_{max}$

- Recalculate zero/pole time constants:
  
  \[
  T = \frac{1}{\omega_{max} \sqrt{\beta}} = \frac{1}{5.5 \sqrt{0.147}} = 0.4742
  \]

  \[
  \beta T = 0.147 \cdot 0.4742 = 0.0697
  \]

- The updated lead compensator transfer function:
  
  \[
  D(s) = 12 \frac{(0.4742s + 1)}{(0.0697s + 1)}
  \]
Lead Example 1 – Step 6

- Crossover frequency has been reduced
  - \( \omega_{PM} = 5.58 \text{ rad/sec} \)

- Phase margin is close to the target
  - \( PM = 58.2^\circ \)

- Dip in phase is apparent, because \( \omega_{max} \) is now placed at point of lower open-loop phase
Lead Example 1 – Step 6

- Overshoot requirement now satisfied
  - $\%OS = 14.7\% < 15\%$

- Low-frequency gain has not been changed, so error requirement is still satisfied

- Design is complete
Lead Compensation – Example 2

- Again, consider the same system

- Design a compensator to satisfy the following
  - \( t_s \approx 1.2 \sec \ (\pm 1\%) \)
  - \( \%OS \approx 10\% \)

- Now, we’ll design a lead compensator to simultaneously adjust closed-loop bandwidth and phase margin
Lead Example 2 – Step 1

- The required damping ratio for 10% overshoot is
  \[
  \zeta = -\frac{\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.5912
  \]

- Given the required damping ratio, calculate the required closed-loop bandwidth to yield the desired settling time
  \[
  j\omega_{BW} = 4.6\frac{\omega_{BW}}{t_s \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}
  \]

  \[
  \omega_{BW} = 7.52 \text{ rad/sec}
  \]

- We’ll initially set the gain, \(K\), to place the crossover frequency, \(\omega_{PM}\), one octave below the desired closed-loop bandwidth
  \[
  \omega_{PM} = \frac{\omega_{BW}}{2} = 3.8 \text{ rad/sec}
  \]
Lead Example 2 – Step 1

- Plot the Bode plot for $K = 1$
  - Determine the loop gain at the desired crossover frequency
    \[ K_{PM} = -23.3 \text{ dB} \]
  - Adjust $K$ so that the loop gain at the desired crossover frequency is 0 dB
    \[ K = \frac{1}{K_{PM}} = 23.3 \text{ dB} = 14.7 \]
Lead Example 2 – Steps 2 & 3

- Generate a Bode plot using the gain value just determined
- Phase margin for the uncompensated system:
  \[ PM_u = 14.9^\circ \]
- Required phase margin to satisfy overshoot requirement:
  \[ PM \approx 100\zeta = 59.1^\circ \]
- Add 10° to account for crossover frequency increase
  \[ PM = 69.1^\circ \]
- Required phase lead from the compensator
  \[ \phi_{max} = PM - PM_u = 54.2^\circ \]
Calculate zero/pole spacing, $\beta$, from required phase lead, $\phi_{max}$

$$\beta = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})} = 0.1040$$

Calculate zero and pole time constants

$$T = \frac{1}{\omega_{max}\sqrt{\beta}} = 0.8228 \text{ sec}$$

$$\beta T = 0.0855 \text{ sec}$$

The resulting lead compensator transfer function:

$$KD(s) = K \frac{(Ts + 1)}{(\beta Ts + 1)}$$

$$KD(s) = 14.7 \frac{(0.8228s + 1)}{(0.0855s + 1)}$$
Lead Example 2 – Step 6

- Bode plot of the compensated system
  - $PM = 49.8^\circ$
  - Substantially below target

- Crossover frequency is well above the desired value
  - $\omega_{PM} = 9.44 \text{ rad/sec}$

- Iteration will likely be required
Lead Example 2 – Step 6

- Overshoot exceeds the specified limit
  - \( \%OS = 19.1^\circ > 10\% \)

- Settling time is faster than required
  - \( t_s = 0.98 \text{ sec} < 1.2 \text{ sec} \)

- Iteration is required
  - Start by reducing the target \( \omega_{PM} \)
Lead Example 2 – Step 6

- Must redesign the compensator to meet specifications
  - Must *increase PM* to reduce overshoot
  - Can afford to *reduce crossover*, $\omega_{PM}$, to improve PM

- Try various combinations of the following
  - Reduce crossover frequency, $\omega_{PM}$
  - Increase compensator zero/pole frequencies, $\omega_{max}$
  - Increase added phase lead, $\phi_{max}$, by reducing $\beta$

- Iteration shows acceptable results for:
  - $\omega_{PM} = 2.4 \text{ rad/sec}$
  - $\omega_{max} = 3.4 \text{ rad/sec}$
  - $\phi_{max} = 52^\circ$
Lead Example 2 – Step 6

- Redesigned lead compensator:
  \[ KD(s) = 6.27 \frac{(0.8542s + 1)}{(0.1013s + 1)} \]

- Phase margin:
  \[ PM = 62^\circ \]

- Crossover frequency:
  \[ \omega_{PM} = 4.84 \text{ rad/sec} \]
Lead Example 2 – Step 6

- Dynamic response requirements are now satisfied
- Overshoot:  
  \[\%OS = 8\%\]
- Settling time:  
  \[t_s = 1.09 \text{ sec}\]
Lead Compensation – Example 2

- Lead compensator adds gain at higher frequencies
  - Increased crossover frequency
  - Faster response time

- Phase added near the crossover frequency
  - Improved phase margin
  - Reduced overshoot
Lead Compensation – Example 2

- Step response improvements:
  - Faster settling time
  - Faster risetime
  - Significantly less overshoot and ringing
Lead-Lag Compensation

- If performance specifications require adjustment of:
  - Bandwidth
  - Phase margin
  - Steady-state error

- Lead-lag compensation may be used

\[
D(s) = \alpha \frac{(T_{lag}s + 1)}{(\alpha T_{lag}s + 1)} \frac{(T_{lead}s + 1)}{(\beta T_{lead}s + 1)}
\]

- Many possible design procedures – one possibility:
  1. Design lag compensation to satisfy steady-state error and phase margin
  2. Add lead compensation to increase bandwidth, while maintaining phase margin