Linear programming: the ultimate practical problem-solving model

What is Linear programming?

- The process of minimizing a linear objective function subject to a finite number of linear equality and inequality constraints.
- The word “programming” is historical and predates computer programming.
- Quintessential tool for optimal allocation of scarce resources, among a number of competing activities.
- Why significant?
  - Widely applicable problem-solving model
  - Dominates world of industry.
  - Fast commercial solvers available: CPLEX, OSL
  - Ranked among most important scientific advances of 20th century

An Example: the Diet Problem

- Design a lowest cost diet that provides sufficient protein, with two choices:
  - steak: 2 units of protein/pound, \$3/pound
  - peanut butter: 1 unit of protein/pound, \$2/pound
- Proper diet needs 4 units protein/day.

Let x = # pounds steak/day in the diet.

Let y = # pounds peanut butter/day in the diet.

Goal: minimize $3x + 2y$

Subject to the constraint:

$$2x + y \geq 4$$
$$x \geq 0, y \geq 0$$

Solving this problem geometrically

Linear Program – Definition

- A linear program is a problem with a set of n variables $x_1, x_2, \dots, x_n$ that has

1. A linear objective function which can be minimized or maximized:

$$\min(or \max)\, c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

2. A set of m linear constraints. A constraint looks like:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq (or \geq, =) b_i$$

Solution to LP

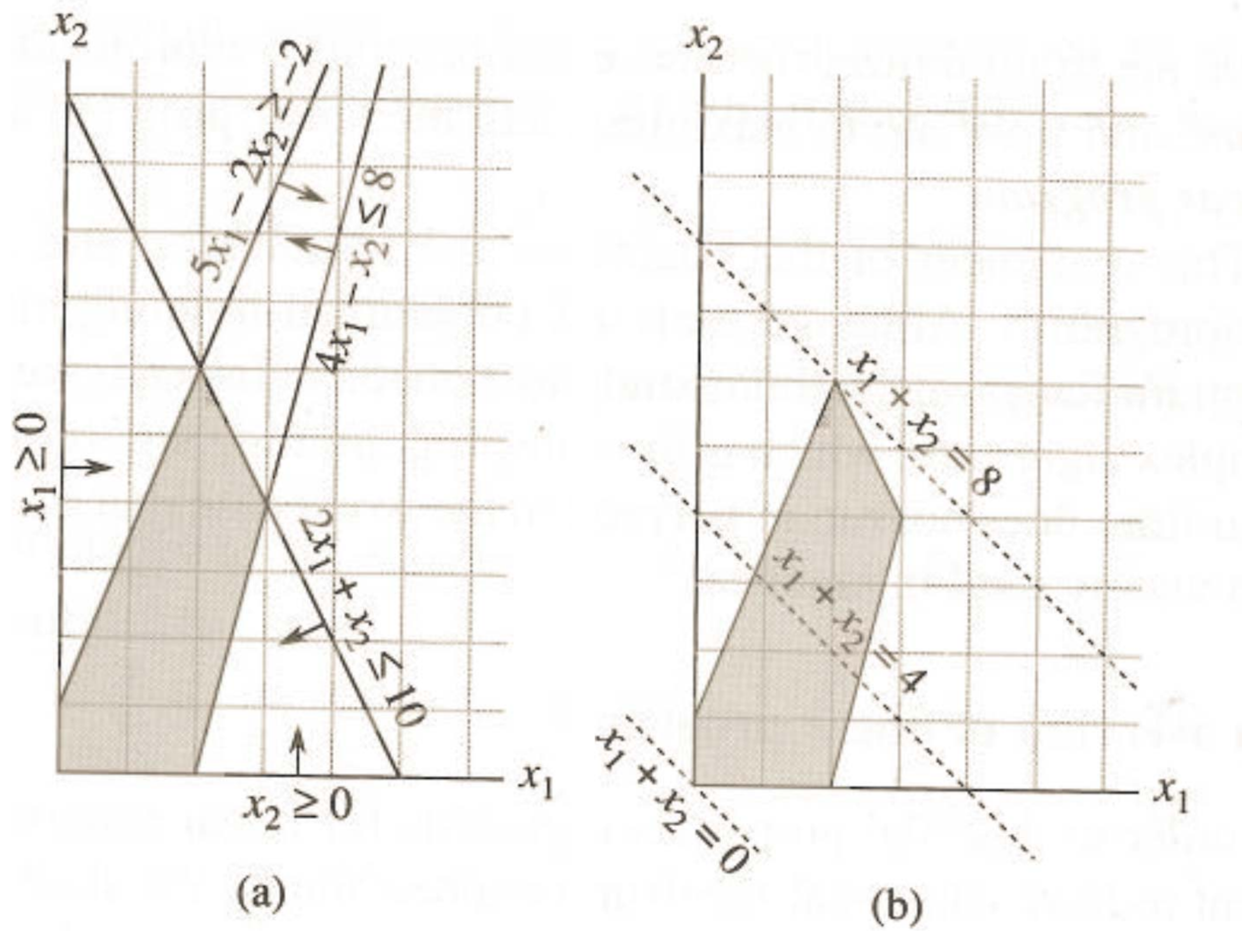
- Each linear inequality divides the n-d space into two half-spaces – only one of them satisfy the constraint
- Feasible set: the set of all points that satisfy all the constraints – the intersection of a set of half-spaces
- Not all linear programs have solutions
  - Infeasible problems – when the feasible set is empty
  - Unbounded – when the objective can achieve arbitrarily high value
- If the feasible set is non-empty and bounded, the optimum must occur at the corner of the feasible set

Example:

$$\max x_1 + x_2$$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1 \geq 0, x_2 \geq 0$$



Feasible set

Contour line of objective

Canonical form of LP

- Objective (must be maximization)

$$\max c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

- Constraints must be $\leq$

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i$$
$$i = 1, \ldots, m$$

- Variables must be non-negative:

$$x_i \geq 0$$

Different forms are equivalent

- a <u>maximization</u> problem can be turned into a <u>minimization</u> problem

$$\max c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

is equivalent to

$$\min -c_1x_1 - c_2x_2 - \cdots - c_nx_n$$

Different forms are equivalent

- Equality constraint can be expressed using two inequality constraints:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

is equivalent to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$$

Different forms are equivalent

- An inequality constraint can be turned into an equality constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

can be expressed by:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s_i = b_i$$
$$s_i \geq 0$$

Different forms are equivalent

- A variable x that is unrestricted in sign can be expressed by a pair of non-negative variables
  - Introduce two nonnegative variables x^+ and x^-
  - Replace x wherever it appears with $x^+ - x^-$

- Any feasible solution x to the original LP can be mapped to a feasible solution of the new LP using $x^+ - x^-$

Example:

$$\max x_1 + x_2$$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1 \geq 0, x_2 \geq 0$$

Turn this problem into a minimization problem with equality constraint

LP formulation: another example

- Bob's bakery sells bagel and muffins.
- To bake a dozen bagels Bob needs
  - 5 cups of flour, 2 eggs, and one cup of sugar.
- To bake a dozen muffins Bob needs
  - 4 cups of flour, 4 eggs and two cups of sugar.
- Bob can sell bagels for $10/dozen and muffins for $12/dozen.
- Bob has 50 cups of flour, 30 eggs and 20 cups of sugar.
- How many bagels and muffins should Bob bake in order to maximize his revenue?

More example of reduction

- Problems that don't look like a linear program sometimes can be <u>reduced</u> to LP
- Example:
  - You are given a set of n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$
  - You want to learn a linear model that predicts y from x accurately: $y = ax + b$
  - In particular, you want to minimize the sum of absolute error:

$$\min_{a,b} \sum_{i=1}^{n} |ax_i + b - y_i|$$

- How can we solve this problem using LP?

Solving linear programs

- LP has been widely used for large practical problems for the 50 + years
- Many solvers available – we can think of them as blackboxes
  - With known algorithm that runs in poly time
- Integer linear programing is NP-complete
  - Variables are limited to take integer values – combinatorial optimization problems
  - Without any known poly time algorithm