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A path integral procedure for the analysis of a noisy nonlinear system

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ABSTRACT: Evolution of the probability density (PD) of the response of a nonlinear system driven by periodic excitation with white noise can be described by a Fokker-Planck equation (FPE). In this study, a path integral solution procedure is employed to solve the FPE numerically to examine some features of chaotic and extreme behavior of a stochastic nonlinear system. The evolution path of the PD is discretized in probability space. For infinitesimally small time segments, the short-time propagator is characterized analytically by a multi-variate Gaussian distribution. In the numerical implementation, the short-time propagator is converted into a transition tensor, which is applied iteratively to obtain the PD at a desired time. Evolution of the PD is examined by sampling the probability density recurrently. The steady-state PD on the Poincaré section is used to portray the periodic and chaotic response attractors. With time-averaged PD as an invariant measure, the probability of large excursions is also examined.

INTRODUCTION

Probability descriptions have been introduced to demonstrate the stochastic properties of noisy nonlinear system responses recently. Meunier and Verga (1988) investigated the behavior of a first order nonlinear system driven by Gaussian white noise. The Fokker-Planck equation (FPE) was derived and solved for the stationary probability density (PD). They concluded that in noisy systems topological concepts become meaningless and the bifurcation phenomenon should be delineated by stochastic representations. Kapitaniak (1988) also solved the FPE to demonstrate the characteristics of marginal PD of nonlinear systems driven by periodic and random excitation. Numerical results show that the marginal PD associated with noisy chaos is governed by a time-varying multi-maxima curve (thus non-stationary). Bulsara *et al.* (1990) investigated the noise effect on the behavior of nonlinear systems through the Lyapunov exponent and PD. Their numerical results demonstrate that the presence of random noise smooths the PD and noise-induced chaos can be also observed. Kunert and Pfeiffer (1991) employed a finite difference procedure to solve the FPE. They used the steady-state joint PD on the Poincaré section to portray the existence of chaotic attractors. Kifer (1989) provided mathematical background for the existence of invariant measure of weakly perturbed attractors. Jung and Hänggi (1990) proposed a time-averaged PD as the invariant measure for deterministic and noisy chaos.

In this paper a detailed derivation of a path integral numerical procedure to solve the FPE associated to a periodically driven nonlinear stochastic system for joint PD is demonstrated. The chaotic and extreme behaviors of this system are examined via the time-varying and steady states of the PD.

GOVERNING EQUATION

The system considered has linear damping and a cubic nonlinear component in the restoring force (Duffing system). The response of this system subjected to a (deterministic) periodic excitation with additive white noise is governed by the following equation

$$\ddot{x}+a\dot{x}+bx+cx^3=A\cos\omega t+\eta(t) \tag{1}$$

where A and ω are the amplitude and frequency of the periodic excitation, respectively, and $\eta(t)$ represents a zero-mean, delta-correlated white noise with intensity κ:

$$\langle\eta(t)\rangle=0;\ \ \langle\eta(t')\eta(t)\rangle=\kappa\delta(t'-t) \tag{2}$$

Standard Form

Equation (1) denotes a system with a time-explicit deterministic driving term in the excitation. This system can be converted into a standard form with explicit stochastic excitation only by introducing two additional state variables to describe the periodic forcing term (Kapitaniak, 1988). Equation (1) thus can be rewritten in a generalized vector form as

$$\dot{X} = F(X)+G(X)\eta(t) \tag{3}$$

with

$$\begin{aligned} X &= [x_1\ x_2\ x_3\ x_4]^{\mathrm{T}} \\ &= [x\ \dot{x}\ A\cos\omega t\ -A\omega\sin\omega t]^{\mathrm{T}} \end{aligned} \tag{4}$$

$$F(X,t)=\begin{Bmatrix} x_2 \\ -ax_2-bx_1-cx_1^3+x_3 \\ x_4 \\ -\omega^2x_3 \end{Bmatrix} \tag{5}$$

$$G(X) = [0\ 1\ 0\ 0]^{\mathrm{T}} \tag{6}$$

Equation (3) is in a standard form that the associated FPE can be derived explicitly.

FOKKER-PLANCK EQUATION

Under the assumption of a Markov process, the evolution of the PD of the system response is characterized by a FPE. A general form of the FPE associated to equation (3) is given by

$$\frac{\partial f(X,t)}{\partial t} = Lf(X,t) \tag{7}$$

with

$$L=\frac{1}{2}\frac{\partial^2}{\partial x_\nu\partial x_\mu}Q_{\nu\mu}(X)-\frac{\partial}{\partial x_\nu}K_\nu(X,t) \qquad \nu,\mu = 1,2,3,4 \tag{8}$$

and

$$K_\nu=F_\nu(X);\quad Q_{\nu\mu}=\kappa G_\nu G_\mu \tag{9}$$

where $f(X,t)$ represents the PD, K_1 ($=x_2$), K_2 ($=-ax_2-bx_1-cx_1^3+x_3$), K_3 ($=x_4$) and K_4 ($=-\omega^2x_3$) are the 4 entries in the drift vector $\mathbf{K}$, and Q_{22} ($=\kappa$) is the only non-zero entry in the 4×4 diffusion matrix $\mathbf{Q}$. Note that periodicity in state variables x_3 and x_4 is implied by equation (4), thus the PD is periodic in time with period $2\pi/\omega$ (thus non-stationary) according to Floquet theorem (Stratonovich, 1967).

PATH INTEGRAL SOLUTION

The path integral solution for solving the response of nonlinear stochastic system has been shown to be systematic and computationally efficient (Naess and Johnsen, 1992). In this section, the solution incorporating the time-explicit periodic excitation is derived in detail.

Short-Time Propagator

A short-time transition from time t to $t+\tau$ can be described by a short-time propagator. The short-time transition can be obtained analytically using a first order approximation to equation (7)

$$f(X,t+\tau)=\left(1+\tau L+O(\tau^2)\right)f(X,t) \tag{10}$$

thus the probability density at the N^{th} step is represented by

$$f(X,t_0+N\tau) \cong (1+\tau L)^N f(X_0,t_0) \cong \exp\left[(N\tau)L\right] f(X_0,t_0) \tag{11}$$

The convergence of equation (11) has been demonstrated by Wissel (1979), and the short-time propagator (Green's function) is given by

$$\begin{aligned} P_{t,\tau}(X'\,|\,X) &= [1+\tau L+O(\tau^2)]\delta(X'-X) \\ &\cong \delta(X'-X)+\tau\{\frac{1}{2}\frac{\partial^2[Q_{\mu\nu}\delta(X'-X)]}{\partial x_\mu \partial x_\nu} \\ &\quad -\frac{\partial}{\partial x_\nu}[K_\nu \delta(X'-X)]\} \end{aligned} \tag{12}$$

where subscripts ν and μ denote the differentiations with respect to x_ν and x_μ ($\nu,\mu=1,2$). A coordinate transformation ($U=X'-X$; $V=V(X',X)$) is introduced to manage the Dirac delta functions, and the short-time propagator can be converted into a Fourier representation

$$P_{t,\tau}(U,V) = \int \frac{d\Omega}{(2\pi)^n}\exp(i\Omega U) \times \tilde{P}_{t,\tau}(\Omega,V) \tag{13}$$

where

$$\begin{aligned} \tilde{P}_{t,\tau}(\Omega,V) &= \int \exp(-i\Omega U)P_{t,\tau}(U,V)dU \\ &= 1+\tau K_\nu^{(\nu)}+\frac{\tau}{2}Q_{\nu\mu}^{(\nu\mu)}+\frac{\tau}{2}\frac{\partial^2}{\partial x_\nu \partial x_\mu}Q_{\nu\mu} \\ &\quad -\tau\frac{\partial}{\partial x_\nu}Q_{\nu\mu}^{(\mu)}-\tau\frac{\partial}{\partial x_\nu}K_\nu \end{aligned} \tag{14}$$

By adopting the approximation described in equation (11), equation (14) becomes

$$\begin{aligned} \tilde{P}_{t,\tau}e^{(i\Omega U)} &\cong \exp\{\frac{\tau}{2}\frac{\partial^2 Q_{\nu\mu}}{\partial x_\nu \partial x_\mu}-\frac{\partial}{\partial x_\nu}[\tau Q_{\nu\mu}^{(\mu)} \\ &\quad +\tau K_\nu - U_\nu]+\tau K_\nu^{(\nu)}+\frac{\tau}{2}Q_{\nu\mu}^{(\nu\mu)}\} \end{aligned} \tag{15}$$

Inserting equation (15) into equation (13) and integrating over Ω with the help of quadratic completion (Wissel, 1979), the short-time propagator is then given by

$$\begin{aligned} P_{t,\tau}(X'\,|\,X) &= (2\pi\tau)^{-\frac{n}{2}}Q^{-\frac{1}{2}}\exp\left\{-\frac{\tau}{2}[Q_{\nu\lambda}^{(\lambda)}\right. \\ &\quad +K_\nu-\frac{x_\nu'-x_\nu}{\tau}]\,Q_{\nu\mu}^{-1}\,[Q_{\mu\rho}^{(\rho)}+K_\mu-\frac{x_\mu'-x_\mu}{\tau}] \\ &\quad \left. +\tau K_\nu^{(\nu)}+\frac{\tau}{2}Q_{\nu\mu}^{(\nu\mu)}\right\} \end{aligned} \tag{16}$$

Thus, the general expression of short-time propagator is in a form of multi-variate Gaussian distribution. However, as implied by equation (6), the only source of randomness, Q_{22}, appears in the second expression. Thus, for equation (3), the multi-variate Gaussian form for the short-time propagator in equation (16) degenerates to a product of one-dimensional Gaussian distribution and Dirac delta functions which denote the statistical dependency between x_1 and x_2, and x_3 and x_4:

$$\begin{aligned} P_{t,\tau}(X'\,|\,X) &= \sqrt{2\pi\tau Q_{22}}\,\exp\{-\frac{\tau}{2Q_{22}}(-ax_2 \\ &\quad -bx_1-cx_1^3+x_3-\frac{x_2'-x_2}{\tau})^2\} \\ &\quad \delta(x_2-\frac{x_1'-x_1}{\tau})\;\delta(x_3+\frac{x_4'-x_4}{\omega^2\tau}) \\ &\quad \delta(x_4-\frac{x_3'-x_3}{\tau}) \end{aligned} \tag{17}$$

Numerical Procedure

Using a multi-dimensional histogram representation for the probability density, the path sum (equation (10)) can be implemented numerically. The probability domain at time t is discretized into a finite number of elements represented by function π:

$$\begin{aligned} P(X,t) &= \sum_{i,j,k,l=1}^{N} \pi(x_1-x_{1i})\,\pi(x_2-x_{2j}) \\ &\quad \pi(x_3-x_{3k})\,\pi(x_4-x_{4l})f(X,t) \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\pi(x_N-x_{NJ})=1 \text{ for } x_N-\frac{\Delta x_{N(J-1)}}{2}\le x_N \\ &\le x_{N(J)}+\frac{\Delta x_{N(J)}}{2}\ ;\quad 0 \text{ otherwise} \\ &\text{with}\quad N=1,2,3,4;\ \ J=i,j,k,l \end{aligned} \tag{19}$$

Thus, the short-time propagator is also discretized into a short-time transition tensor $T_{ijkl,mnop}(\tau)$. Subscripts i,j,k,l and m,n,o,p represent the signatures of the elements of the 4-D probability domain at the pre- and post-state, respectively. Note again that the last two Dirac delta functions of the short-time propagator (equation (17)) implies that state variables x_3 and x_4 are statistically independent of the randomness, denoted by Q_{22}. The independence can be used to reduce the dimension of the transition tensor from 4×4 to 2×2 ($T_{ij,kl}(\tau)$). Subscripts i,j and k,l of the dimension-reduced transition tensor ($T_{ij,kl}$) represent the signatures of the elements of the pre- and post- x_1-x_2 probability domains, respectively. The corresponding dimension-reduced short-time propagator is given by

$$P_{t,\tau}(X'|X) = \sqrt{2\pi\tau Q_{22}}\ \exp\{-\frac{\tau}{2Q_{22}}(-ax_2 - bx_1 - cx_1^3 + x_3 - \frac{x_2' - x_2}{\tau})^2\}\ \delta(x_2 - \frac{x_1' - x_1}{\tau}) \tag{20}$$

and the relationship between state variables x_3 and x_4 can be expressed explicitly as

$$\begin{aligned} x_3' &= x_3 + \tau x_4 \\ x_4' &= x_4 - \omega^2 \tau x_3 \end{aligned} \tag{21}$$

Thus a short-time propagation for each element at the pre-state can be numerically implemented by determining the associated most probable position in the phase space and, with which as the mean, the random response following a Gaussian distribution. The most probable phase position after a short-time propagation for each element is deterministically computed from the drift coefficients, and the random response is Gaussian distributed and outstretched in x_2 direction. Thus the PD at time $t+\tau$ can be obtained by summing all the probability mass propagated from time t (and normalizing afterward):

$$P_{ij}(t+\tau) = T_{ij,kl}(\tau) P_{kl}(t) \tag{22}$$

where repeat indices indicate summation, and the transition tensor is given by

$$T_{ij,kl}(\tau) = \frac{2^2}{(\Delta x_{1(i-1)} + \Delta x_{1(i)})(\Delta x_{2(j-1)} + \Delta x_{2(j)})} \int_{x_{1(i)} - \frac{\Delta x_{1(i-1)}}{2}}^{x_{1(i)} + \frac{\Delta x_{1(i)}}{2}} dx_1' \int_{x_{2(j)} - \frac{\Delta x_{2(j-1)}}{2}}^{x_{2(j)} + \frac{\Delta x_{2(j)}}{2}} dx_2' \int_{x_{1(k)}) - \frac{\Delta x_{1(k-1)}}{2}}^{x_{1(k)} + \frac{\Delta x_{1(k)}}{2}} dx_1 \int_{x_{2(l)} - \frac{\Delta x_{2(l-1)}}{2}}^{x_{2(l)} + \frac{\Delta x_{2(l)}}{2}} dx_2\ P_{t,\tau}(X'|X) \tag{23}$$

The PD at the desired time can be obtained by applying the short-time transition in equation (22) iteratively. Note that, because of the smooth nature of the PD for purely random response, a B-spline smoothing technique was applied by Johnsen and Naess (1991) to save computational time. However, due to the fact that the marginal PD of a chaotic response is governed by a jagged curve (Kapitaniak, 1988), to preserve the fractal nature of the chaotic response, no smoothing techniques are employed in this study.

To obtain accurate numerical results, the grid size of the discretized probability domain has to be sufficient small. On the other hand, the computational time will increase geometrically with the number of elements. To improve numerical accuracy and reduce computational efforts, the concept of moving boundary is employed. For the numerical results presented in this study, the initial conditions are assumed deterministic. They are represented by a product of two Dirac delta functions

$$P(X, t_0) = \delta(x_1 - x_{10})\,\delta(x_2 - x_{20}) \tag{24}$$

which resembles a dot with virtually zero area in the phase space (see Fig. 1a). Thus the probability domain (pre-state) is set just large enough to enclose this dot, and the domain (post-state) after a short-time propagation is pre-set by the spreading behavior indicated by Q_{22} in equation (20). With the adjustment of the boundary for the probability domain, the numerical procedure for the path integral solution is computationally efficient.

Error Reduction

Discretization-induced errors in the numerical procedure are closely related to the size of time segment and the grid size of probability domains, and they can be minimized by normalizing both the short-time propagator and the PD at each time step:

$$\sum_i \frac{1}{2} T_{ij}(\tau)(\Delta X_i + \Delta X_{i-1}) = \frac{1}{2}(\Delta X_j + \Delta X_{j-1}) \times A_j(\Delta X_i, \tau); \quad T_{ij}^{new} \rightarrow \frac{T_{ij}^{old}}{A_j} \tag{25}$$

and

$$\sum_i P_i(X,t)\Delta X_i = 1 + \epsilon(\Delta X_i, \tau) \Rightarrow P_i(X,t) = \frac{1}{1+\epsilon(\Delta X_i,\tau)} P_i(X,t) \tag{26}$$

where subscripts i,j indicate the pre- and post-state, respectively.

RESPONSE DENSITY EVOLUTION

Evolution of the PD can be illustrated by sampling the density recurrently. The slightly disturbed chaotic system is excited with deterministic initial condition at (0,0) (Fig. 1a). The PD starts spreading and tends to cover the chaotic domain after two cycles of the forcing period (Fig. 1b). The PD becomes invariant with time after about 20 cycles (Fig. 1c), and the chaotic attractor is clearly portrayed.

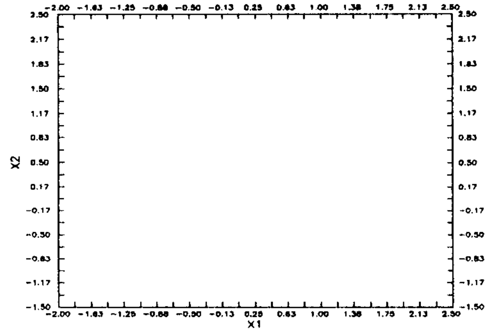
COEXISTING ATTRACTORS

The PD describes the global behavior of the system, hence coexisting response attractors and their relative strengths can be demonstrated by the steady-state PD. As shown in Fig. 2b, the PD is considerably concentrated in the periodic domain. Thus, the periodic attractor is relatively stronger compared to the chaotic attractor.

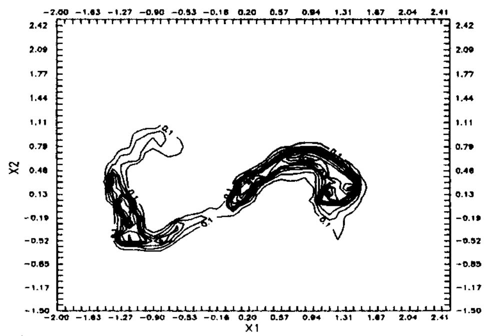
INVARIANT MEASURE

Periodicity in the PD can be removed by time averaging to form a time-averaged density. This density can serve as an invariant measure for both deterministic and noisy chaos (Jung and Hänggi, 1990)

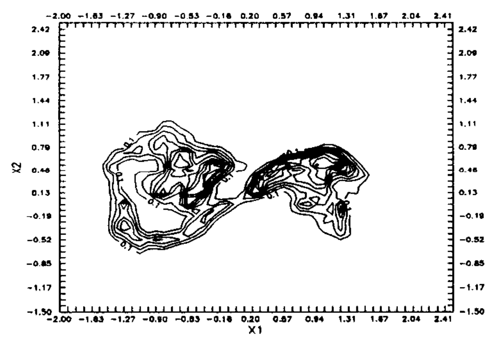
$$P_{av}(x_1,x_2) = \frac{1}{t_n}\int_0^{t_n} P(x_1,x_2,t)\,dt \tag{27}$$



(a)



(b)



(c)

Fig. 1 Evolution of probability density

where $t_n \gg 0$. The invariant measure on x_1 can be defined similarly as

$$P_{av}(x_1) = \int_{-\infty}^{\infty} P_{av}(x_1, x_2) dx_2 \tag{28}$$

The accuracy of this measure is calibrated by analytical solutions to purely random responses (no periodic excitation component) (Johnsen and Naess, 1991). Thus, the time-average PD is then used as an invariant measure for the chaotic response to estimate the probability of large excursions (extreme behavior).

EXTREME ESTIMATES

The presence of random noise causes smoothness and out-stretched tails in PD, which imply that, although at low probability level, large excursion could occur in the response. As stated, the time-average PD is used as a measure for extreme estimates.

By employing Rice's formula (Spanos and Roberts, 1990) the mean up-crossing frequency can be evaluated:

$$\mu^{+}(x_1 = x_d) = \int_{0}^{\infty} x_2 P_{av}(x_d, x_2) dx_2 \tag{29}$$

Adopting the assumption of statistically independent up-crossings (which leads to Poisson distributed crossing events), the asymptotic approximation of probability for x_1 exceeding displacement x_d during time t_n is given by:

$$P_F(x_d, t_n) = 1 - \exp[-\mu^{+}(x_d) t_n] \tag{30}$$

Figure 3a shows the time-averaged PD for the noisy chaotic attractors with noise intensity at 0.02 (dashed line) and 0.07 (solid line). The corresponding probability of large excursions (t_n = 10) is shown in Fig. 3b. With this measure, the design incorporating extreme excursion induced failure in a noisy chaotic system may be derived. Figures 3a and b demonstrate that the probability of large excursions is increased as the noise intensity increases.

CONCLUDING REMARKS

A stochastic nonlinear system may exhibit noisy chaotic response. The evolution of the

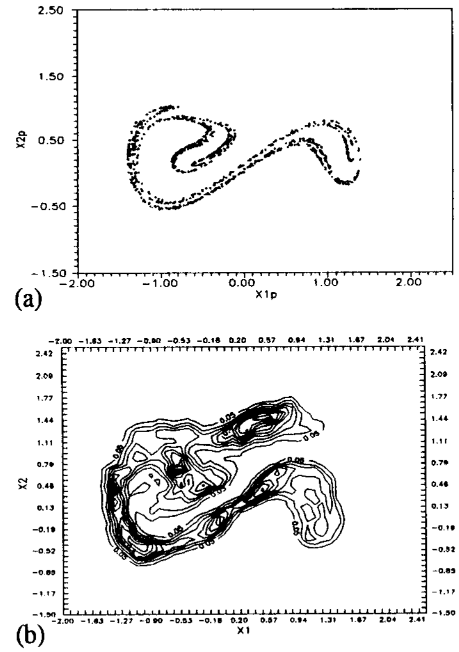


Fig. 2 Coexisting chaotic and periodic attractors: a) Poincaré map; b) density

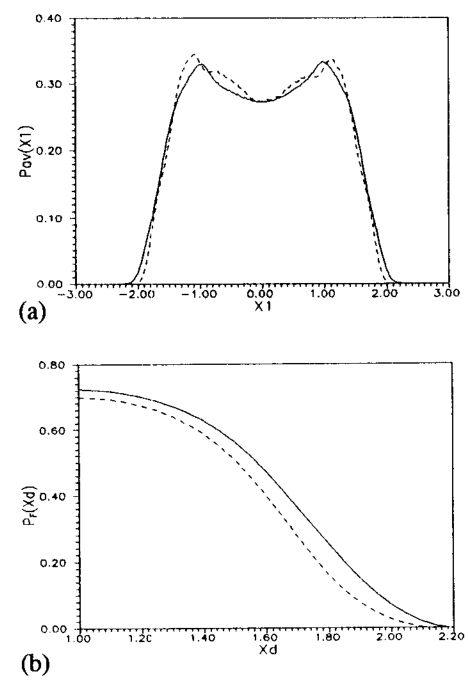


Fig. 3 a) Invariant measure; b) Probability of large excursions probability density (PD) of the response is governed by the Fokker-Planck equation (FPE). The path integral algorithm is used in this study to numerically solve the FPE. This numerical procedure is systematic and computationally efficient. The resulting steady-state PD can portray the (co-) existing response attractors, and govern their relative strengths. Using the time-averaged PD as an invariant measure, the probability of large excursions for a noisy chaotic system is estimated. Numerical results show that the probability of large excursions is increased as the noise intensity increases.

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