Dynamics of Structures on Two-Spring Foundation Allowed to Uplift

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Abstract: Investigated in this paper is the dynamics of structures with their foundation mat supported only through gravity and thus permitted to uplift from the supporting system. In its fixed base condition the structure is idealized as a single-degree-of-freedom system attached to a rigid foundation mat, which is supported at each edge by a spring-damper element. Analytical expressions are presented for the free vibration response of the system and the effects of foundation uplift are examined. An effective numerical procedure, based on expressions of the Rayleigh-Ritz concept, to evaluate the structural response to earthquakes is presented. Based on the response spectra presented it is shown that foundation mat uplift has the effect of reducing the base shear for short-period structures, with these reductions being especially significant for the more slender structures.

Introduction

Standard earthquake analysis of buildings under the assumption that the foundation and soil are firmly bonded, often produce a base overturning moment that exceeds the available overturning resistance due to gravity loads, which implies that a portion of the foundation would intermittently uplift during an earthquake. By investigating the dynamics of rocking rigid blocks, Housner was the first to recognize the correlation between foundation uplift and the good performance of seemingly unstable structures during earthquakes (3). Idealizing the structure as a one-degree-of-freedom oscillator, Meek analytically investigated the effects of foundation uplift on the earthquake response of a flexible structure (5). Meek concluded that foundation uplift leads to reduction in the structural deformation. Later, Meek extended his study to multistory, braced-core buildings (6), and found that significant reduction in structural deformation can again be obtained by permitting uplift of the rigid base of the core. Experiments on steel building frames with columns permitted to lift-off during vibration, conducted on the Berkeley shaking table (2,4), demonstrated that allowing column uplift under an extreme earthquake would enhance the chances of a structure surviving in a functional condition, with damage held to a minimum. The flexibility of the supporting soil was ignored in these analytical studies (5,6); it was not incorporated in the shaking table tests either (2,4).

Recently, Psycharis (7) studied the effects of foundation uplift on structural response using two simple models to represent foundation flexibility and energy dissipation: (1) Two spring-damper elements symmetrically placed under the base of the structure; and (2) Winkler foundation with spring-damper elements distributed over the base of the structure. The analytical parts of his study included several aspects of the dynamics of rigid blocks as well as flexible structures. Also he numerically examined the effects of base uplift on the dynamic response of one example single-story structure and one multistory superstructure. Based on these results, he concluded that "for flexible superstructures, it seems that uplift always increases the angle of rotation of the foundation mat but the effects on the structural deflection and the resulting stresses are not clear. In general, it cannot be concluded whether uplift is beneficial to the structure or not, since this depends on the parameters of the system and the characteristics of the excitation."

The objective of this paper is to develop a better understanding of the effects of transient foundation uplift on building response, considering a wide range of the important system parameters. For this purpose it is appropriate to choose structural idealizations that are relatively simple but realistic in the sense that they incorporate only the most important features of foundation uplift. In its fixed base condition, the structure is idealized as a single-degree-of-freedom system attached to a rigid foundation mat which is flexibly supported. The flexibility and damping of the supporting soil is represented by two spring-damper (in parallel) elements, one at each edge of the foundation mat.

System Considered

It is desirable to begin with the simplest possible structural system, so for this purpose we consider the idealized representation of a one-story structure shown in Fig. 1(*a*). It is a linear structure of mass m, lateral stiffness k and lateral damping c, which is supported through the foundation mat of mass m_o resting on two spring-damper elements, one at each edge of the foundation mat, connected to the base which is as-

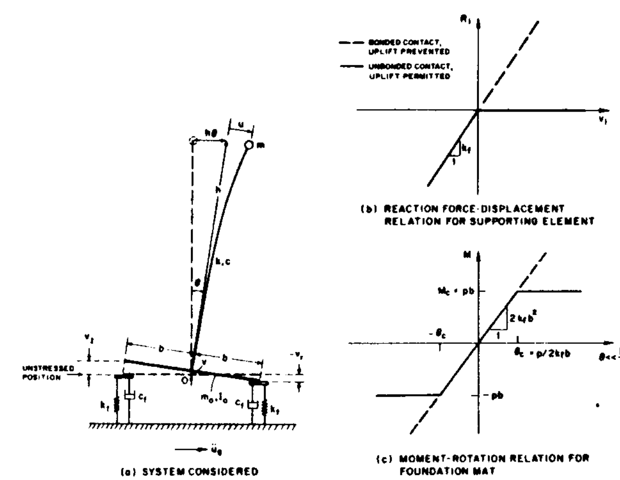

FIG. 1.—Flexible Structure on Two Spring-Damper Element Foundation and Its Properties

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Note.—Discussion open until December 1, 1984. To extend the closing date one month, a written request must be filed with the ASCE Manager of Technical and Professional Publications. The manuscript for this paper was submitted for review and possible publication on August 23, 1983. This paper is part of the *Journal of Engineering Mechanics*, Vol. 110, No. 7, July, 1984. ©ASCE, ISSN 0733-9399/84/0007-1124/$01.00. Paper No. 18986.

sumed to be rigid. The columns are assumed to be massless and axially inextensible, the foundation mat is idealized as a rigid rectangular plate of negligible thickness with uniformly distributed mass, and it is presumed that horizontal slippage between the mat and supporting elements is not possible. The stiffness and damping coefficients of the foundation model are assumed constant, independent of displacement amplitude or excitation frequency. Thus, the frequency dependence of these coefficients, as for a viscoelastic half space (9), is not recognized; nor is the strain dependence of these coefficients for soils (8) considered in this study.

In previous studies, the impact between the foundation mat of a flexible structure (5), or the base of a rocking rigid block (3), and the rigid supporting soil was idealized as a perfectly inelastic collision with the vertical velocity and associated kinetic energy of the structural mass dissipated instantaneously. However, in the system idealization selected here, the energy is gradually dissipated through the viscous damping mechanism of the foundation. But if the foundation springs are relatively stiff, the typical case for buildings and soils, the energy dissipation occurs within a very short duration. Thus, the two types of energy dissipating mechanisms would be expected to similarly affect structural response. Psycharis also concluded that the impact effect can be modeled well by the simpler viscous damping mechanism (7).

Prior to the dynamic excitation, the foundation mat rests on the spring-damper elements only through gravity and is not bonded to these supporting elements. Thus, a supporting element can provide an upward reaction force to the foundation mat but not a downward pull. During vibration of the system this upward reaction force will vary with time. At any instant of time the total reaction force—which for a supporting element consisting of a spring and damper in parallel is the sum of the elastic and damping forces—reaches zero, one edge of foundation mat is in the condition of incipient uplift from the supporting element.

In particular, if the damper is absent, the relation between the reaction force and displacement of an edge i (left or right) of the foundation mat is shown in Fig. 1(*b*). The upward reaction force is related linearly to the downward displacement of the foundation mat through the spring stiffness k_f, but no reaction force is developed if the displacement is upward; the displacements are measured from the unstressed position of the springs. If the foundation mat were bonded to the supporting elements the force-displacement relation will be linear, as shown in Fig. 1(*b*), valid for downward as well as upward displacements.

Consider the foundation mat and its supporting elements without the superstructure with a static force p acting in the downward direction at its center of gravity (c.g). The relation between the static moment M applied at the c.g and the resulting foundation-mat rotation θ, limited to angles much smaller than the slenderness ratio b/h, is shown in Fig. 1(*c*) for unbonded as well as bonded conditions. If the mat is not bonded to the supporting elements the M-θ relation is linear until the foundation mat uplifts from one of the supporting elements; thereafter no additional moment can be developed. Uplift occurs when the rotation reaches $\theta_c = p/2k_f b$ with the corresponding moment $M_c = pb$. The downward force is $p = (m + m_o)g$, the combined weight of the superstructure and foundation mat, prior to any dynamic excitation, but would vary with time during vibration.

Next consider the entire structural system with a gradually increasing force f_s applied in the lateral direction. If the foundation mat is bonded to the supporting elements, which along with the structure have linear properties, the lateral force can increase beyond limit if the overturning effects of gravity forces are neglected. However, if the mat is not bonded to the supporting elements, one edge of the foundation mat is at incipient uplift when the lateral force reaches $f_{sc} = (m + m_o)gb/h$. Thus, the maximum base shear that can be developed under the action of static forces is

$$\mathbf{V}_c = (m + m_o)g\frac{b}{h} \qquad (1a)$$

The structural deformation associated with this base shear is

$$u_c = \frac{(m + m_o)g}{k}\frac{b}{h} \qquad (1b)$$

and at incipient uplift, the foundation-mat rotation

$$\theta_c = \frac{(m + m_o)g}{2k_f b} \qquad (1c)$$

which is consistent with the preceding paragraph and Fig. 1(*c*).

The base excitation is specified by the horizontal ground motion with displacement $u_g(t)$ and acceleration $\ddot{u}_g(t)$. The vertical component of ground motion is not considered in this study. Under the influence of this excitation the foundation mat would rotate through an angle $\theta(t)$ and undergo a vertical displacement $v(t)$, defined at its c.g relative to the unstressed position. Prior to dynamic excitation, the vertical displacement is $v_s = (m + m_o)g/2k_f$, the static displacement due to the total weight of the superstructure and foundation mat. During the dynamic excitation, the vertical displacement v will remain constant at the initial static value if the foundation mat is bonded to the supporting elements, but it will vary with time in the unbonded case. The displaced configuration of the structure at any instant of time can be defined by the deformation $u(t)$, foundation-mat rotation $\theta(t)$, and vertical displacement $v(t)$ at the center of gravity of the foundation mat.

Equations of Motion

The differential equations governing the small-amplitude motion of the system of Fig. 1(*a*) can be derived by considering the lateral equilibrium of forces acting on the structural mass m, and moment and vertical equilibrium of forces acting on the entire system (see Appendix B, Ref. 10). Assuming that the structural-foundation system and excitation are such that the amplitudes of the resulting displacement and rotation responses are small so that sin θ and cos θ may be approximated by θ and 1, respectively, these equations may be expressed as

$$m\ddot{u} + m(h\ddot{\theta}) + c\dot{u} + ku = -m\ddot{u}_g(t) \qquad (2a)$$

$$\frac{m_o b^2}{3h^2}(h\ddot{\theta}) - c\dot{u} + \epsilon_1 c_f \frac{b^2}{h^2}(h\dot{\theta}) + \epsilon_2 c_f \frac{b}{h}\dot{v} - ku$$

$$+ \epsilon_1 k_f \frac{b^2}{h^2}(h\theta) + \epsilon_2 k_f \frac{b}{h} v = 0 \quad \text{(2b)}$$

$$(m + m_o)\ddot{v} + \epsilon_1 c_f \dot{v} + \epsilon_2 c_f \frac{b}{h}(h\dot{\theta}) + \epsilon_1 k_f v + \epsilon_2 k_f \frac{b}{h}(h\theta) = -(m + m_o)g \quad \text{(2c)}$$

in which ϵ_1 and ϵ_2 depend on whether one or both edges of the foundation mat are in contact with the supporting elements; $\epsilon_1 = 2$ and $\epsilon_2 = 0$ for contact at both edges, $\epsilon_1 = 1$ and $\epsilon_2 = \pm 1$ if right or left edge is uplifted. A supporting spring-damper element provides an upward reaction force to the foundation mat

$$R_i(t) = -k_f v_i(t) - c_f \dot{v}_i(t), \quad i = l, r \quad \text{(3a)}$$

in which $v_i(t) = v(t) \pm b\theta(t), \quad i = l, r$ (3b)

As mentioned earlier, a downward reaction force cannot be developed because the foundation mat is not bonded to the supporting elements, i.e. $R_i(t) \geq 0$, $i = l, r$. If, at some instant of time, this condition is not satisfied at the left or right edge of the foundation mat, i.e. $R_i(t) < 0$, $i = l, r$ that particular edge uplifts from the supporting element which no longer provides any reaction to the foundation mat. Because the spring-damper element cannot extend above its initial unstressed position, i.e. $v_i(t) \leq 0$, $i = l, r$, an edge of the foundation mat would uplift at the instant of time this condition is not satisfied. These two uplift criteria are equivalent if each supporting element included only a spring.

The earthquake response of the system depends on the following dimensionless parameters: (1) $\omega = \sqrt{k/m}$, the natural frequency of the rigidly supported structure; (2) $\xi = c/2m\omega$, the damping ratio of the rigidly supported structure; (3) $\beta = \omega_v/\omega$, where $\omega_v = \sqrt{2k_f/(m + m_o)}$ is the vertical vibration frequency of the system of Fig. 1(*a*) with its foundation mat bonded to the supporting elements; (4) $\xi_v = 2c_f/2(m + m_o)\omega_v$, the damping ratio in vertical vibration of the system of Fig. 1(*a*) with its foundation mat bonded to the supporting elements; (5) $\alpha = h/b$, a slenderness-ratio parameter; and (6) $\gamma = m_o/m$, the ratio of foundation mass to superstructure mass.

When the equations of motion and uplift criterion are expressed in terms of these dimensionless parameters, for a given excitation, it is observed that, under the small-amplitude assumption, the foundation-mat rotation θ depends on the width of the foundation mat; however, the structural deformation u, the product of the foundation-mat rotation and half base width θb, and the vertical displacement v, do not depend on the size of the structure. If we were to consider the large-amplitude motion of the structure including the possibility of its overturning, the size parameter h would play an important role, just as in the dynamics of rigid blocks (3). However, the foundations of most buildings are expected to undergo only small rotations and uplift displacements. At these small-amplitude motions the deformation response of the system depends on the h/b ratio but not separately on h.

The equations of motion for the system of Fig. 1(*a*) are nonlinear as indicated by the dependence of the coefficients ϵ_1 and ϵ_2 on whether both edges of the foundation mat are in contact with the supporting elements, the left edge is uplifted, or the right edge is uplifted. However, for each of the three contact conditions the governing equations are linear, but they depend on the contact condition. Thus, during the time duration that one particular contact condition is valid, the system behaves as a linear system, but the vibration properties of the linear system are different for the three contact conditions. Thus the dynamic response of the nonlinear system of Fig. 1(*a*) can be viewed as the sequential response of three linear systems.

The equations of motion can be specialized for the undamped system with massless foundation mat by substituting $m_o = 0$, and $c = c_f = 0$. In particular, the inertia and damping terms in Eq. 2*b* are zero and, following the usual approach to static condensation, θ can be expressed in terms of u and v from

$$-ku + \epsilon_1 k_f \frac{b^2}{h^2}(h\theta) + \epsilon_2 k_f \frac{b}{h} v = 0 \quad \text{(4)}$$

and substituted into Eqs. 2*a* and 2*c*. The reduced system consists of the resulting two differential equations in the two unknown "dynamic" degrees-of-freedom. As presented later, this reduced system of equations provides a basis for approximate analysis of systems with damping and foundation-mat mass, i.e. $c \neq 0$, $c_f \neq 0$, and $m_o \neq 0$.

Natural Vibration Frequencies and Modes

For the special case of $\gamma = 0$, the natural frequencies and mode shapes of the system of Fig. 1(*a*) can be determined by solving the eigenvalue problems associated with the reduced version of Eq. 2. The resulting two-degree-of-freedom eigenvalue problem is solved for the three linear systems corresponding to the three sets of values for ϵ_1 and ϵ_2. The natural frequencies of the linear system with both edges of the foundation mat in contact with the supporting elements are

$$\omega_1 = \omega\bar{\omega}_1; \quad \omega_2 = \omega_v = \omega\beta \quad \text{(5a)}$$

in which $\bar{\omega}_1 = \sqrt{\dfrac{\beta^2}{\alpha^2 + \beta^2}}$ (5b)

and the corresponding mode shapes are

$$\phi_1^T = \langle \bar{\omega}_1^2 \ (1 - \bar{\omega}_1^2) \ 0 \rangle \quad \text{(6a)}$$

$$\phi_2^T = \langle 0 \ 0 \ 1 \rangle \quad \text{(6b)}$$

where the three terms in each mode shape are ordered to correspond to $\mathbf{r}^T = \langle u \ h\theta \ v \rangle$. As shown in Fig. 2(*a*), the system vibrating in the second mode undergoes uncoupled vertical motion, without lateral deformation of the structure or rotation of the foundation mat, at the natural frequency ω_v defined earlier. The motion of the system vibrating in the first mode consists of lateral deformation of the structure and rotation

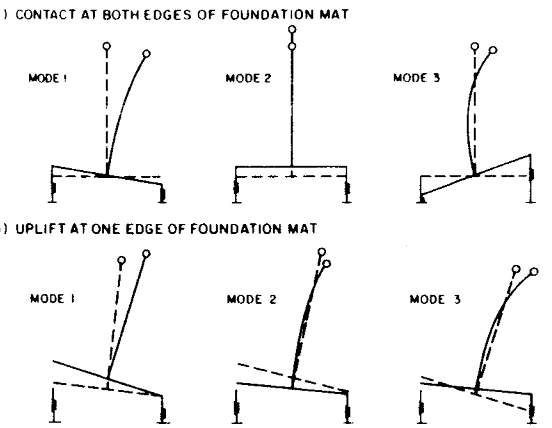

FIG. 2.—Free Vibration Mode Shapes for Two Cases: (*a*) Foundation Mat in Contact with Supporting Elements at Both Edges; and (*b*) One Edge of Foundation Mat Uplifted

of the foundation mat without any vertical displacement of its c.g (relative to its static equilibrium position).

The natural frequencies of the linear system with uplift at one edge (left or right edge) of the foundation mat are

$$\lambda_1 = 0; \quad \lambda_2 = \omega\bar{\lambda}_2 \qquad (7a)$$

in which $$\bar{\lambda}_2 = \sqrt{\frac{(1+\alpha^2)\beta^2}{2\alpha^2+\beta^2}} \qquad (7b)$$

and the corresponding mode shapes are

$$\psi_1^T = \left\langle 0 \;\; 1 \;\; \pm\frac{1}{\alpha} \right\rangle \qquad (8a)$$

$$\psi_2^T = \langle \bar{\lambda}_2^2 \;\; (1-\bar{\lambda}_2^2) \;\; \mp\alpha \rangle \qquad (8b)$$

Eqs. 7 and 8 are applicable to the system with uplift at left or right edge of the foundation mat. Wherever both algebraic signs appear simultaneously the upper sign applies to uplift at the left edge and the lower sign to uplift at the right edge. The mode shapes described by Eq. 8 are plotted in Fig. 2(*b*) relative to an initial reference position (shown in dashed lines) which is somewhat arbitrary. The system vibrating in the first mode undergoes rigid body rotation about one edge of the foundation mat, whereas in the second mode motion occurs in all three degrees of freedom: lateral deformation of the structure, vertical displacement and rotation of the foundation mat. If the frequency ratio β exceeds the slenderness-ratio parameter α, which in turn is much larger than unity, it can be shown that $\bar{\lambda}_2$ is close to α, the value determined directly for the structure of Fig. 1(*a*) but supported directly on the rigid base without the spring-damper elements (5).

For the general case with some foundation-mat mass ($\gamma \neq 0$), the natural frequencies and mode shapes can be determined by solving the eigenvalue problem associated with Eq. 2. The three-degree-of-freedom eigenvalue problem was solved for the three linear systems corresponding to the three sets of values for ϵ_1 and ϵ_2 (see Appendix C, Ref. 10).

For the linear system with both edges of the foundation-mat in contact with the supporting elements, the first two natural frequencies and modes are described by Eq. 5*a* and Fig. 2(*a*) even when the foundation mat has some mass. However, now $\bar{\omega}_1$ depends also on γ and is no longer described by Eq. 5*b*. Because the natural frequency and the shape of the first mode involves $\bar{\omega}_1$ (Eqs. 6*a* and 7*a*), they also depend on γ. The influence of γ on $\bar{\omega}_1$ decreases monotonically as γ becomes smaller. For a fixed value of the frequency ratio β, the natural frequency and shape of the second mode is independent of the mass ratio γ. A third vibration mode appears (Fig. 2(*a*)) when the foundation mat has some mass. Just like the first mode, the third mode includes lateral deformation of the structure and rotation of the foundation mat without any vertical displacement of its c.g (relative to its static equilibrium position). It can be shown through numerical results and a mathematical limiting process that the natural frequency of the third mode is at least an order of magnitude higher than the other two frequencies, and it tends to infinity as the mass of the foundation mat approaches zero (see Appendix C, Ref. 10).

For the linear system with uplift at one edge of the foundation mat, the first two natural frequencies and modes are described by Eqs. 7 and 8 and Fig. 2(*b*), even when the foundation mat has some mass. However, $\bar{\lambda}_2$ is no longer described by Eq. 7*b* and depends on γ. The effect of mass ratio γ on the natural frequency and shape of the second mode decreases monotonically with γ. The natural frequency and shape of the first mode, which is a rigid body mode, is independent of the mass ratio γ. The third vibration mode, which appears when the foundation has some mass, includes displacements in all the three degrees of freedom just as in the second mode. Just as in the earlier case, the natural frequency of the third mode is at least an order of magnitude higher than the second mode frequency and it tends to infinity as the foundation-mat mass approaches zero (see Appendix C, Ref. 10).

Free Vibration Response

Undamped System.—Starting with the initial state of the structural system of Fig. 1(*a*), wherein both edges of the foundation mat are in contact with the supporting elements, the response of the system with massless foundation mat to any initial displacements and velocities can be determined as the superposition of the free vibration responses in the two modes of vibrations ϕ_1, and ϕ_2. In particular, if an initial velocity $\dot{x}(0)$ is imparted in the lateral direction to the mass m of the superstructure, the second vibration mode will not be excited and the response will be entirely due to the first mode. The structural displacement vector $\mathbf{r}^T = \langle u \;\; h\theta \;\; v \rangle$ is then described by

$$\mathbf{r}(t) = \frac{\dot{x}(0)}{\omega_1}\phi_1 \sin \omega_1 t - v_s\phi_2 \qquad (9)$$

provided the motion is small enough that both edges of the foundation mat remain in contact with the supporting elements. From Eq. 9, the structural displacements at the point of incipient uplift of one edge of the foundation mat are

$$u_c = \frac{g}{\alpha\omega^2}; \quad h\theta_c = \frac{\alpha g}{\beta^2\omega^2} \quad \text{(10)}$$

These displacements are same as those presented in Eq. 1, specialized for massless foundation mat, which were obtained from static considerations.

The critical velocity $\dot{x}(0)_c$ is defined as the initial velocity that results in the maximum displacements $\mathbf{r}(t)$, from Eq. 9, equal to the displacements given by Eq. 10. It can be shown that

$$\dot{x}(0)_c = \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha\beta}\frac{g}{\omega} \quad \text{(11)}$$

and it depends on the slenderness-ratio parameter α and frequency ratio β. For purposes of the subsequent discussion it is useful to introduce the normalized initial velocity $\bar{\dot{x}}(0) = \dot{x}(0)/\dot{x}(0)_c$. If the initial velocity is less than the critical velocity, i.e., $\bar{\dot{x}}(0) < 1$, both edges of the foundation mat will remain in contact with the supporting elements and no uplift will occur; the free vibration response will be described by Eq. 9.

If the initial velocity exceeds the critical velocity, i.e., $\bar{\dot{x}}(0) > 1$, Eq. 9 will be valid only until the first uplift occurs. One edge of the foundation mat will uplift at the time instant $u(t)$ given by Eq. 9 reaches the u_c value of Eq. 10. During the time duration that uplift of this edge continues, the displacement response can be expressed as

$$\mathbf{r}(\bar{t}) = \left[\mp\frac{1}{2}\frac{\alpha}{1+\alpha^2}g\bar{t}^2 + C_1\bar{t} + C_3\right]\psi_1$$

$$+ \left[C_2\sin(\lambda_2\bar{t} + \delta) + \frac{1}{\lambda_2^2}\left(\frac{\alpha}{1+\alpha^2}\right)g\right]\psi_2 \quad \text{(12)}$$

wherein $\bar{t}$ is the time measured from the onset of uplift and, where two algebraic signs appear, the upper sign is to be used if the left edge of the foundation mat uplifts and the lower sign applies to uplift of the right edge. The four constants C_1, C_2, C_3 and δ can be determined from the displacements and velocities of the system at the onset of uplift, by transforming them to modal coordinates.

After vibration of the structure for some time with one edge of the foundation mat uplifted, as described by Eq. 12, the foundation mat will re-establish contact with supporting elements at both its edges. Measuring time t' from this instant, the response of the structure during the time span it continues to vibrate without uplift is given by

$$\mathbf{r}(t) = D_1\phi_1\sin(\omega_1 t' + \delta_1) + D_2\phi_2\sin(\omega_2 t' + \delta_2) - v_s\phi_2 \quad \text{(13)}$$

Both vibration modes may now contribute to the response, in contrast to Eq. 9 wherein the second vibration mode did not appear. The four constants D_1, D_2, δ_1 and δ_2 can be determined by standard procedures

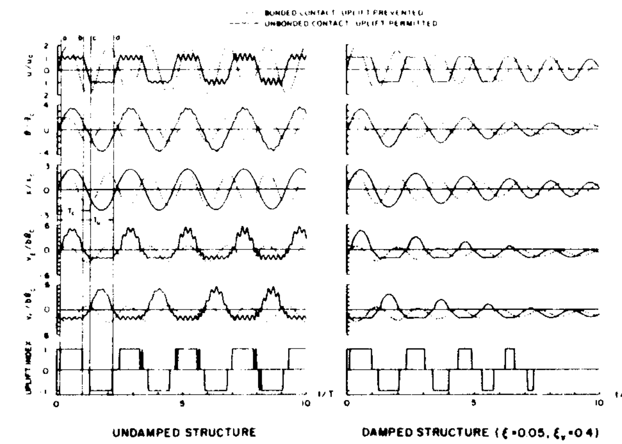

FIG. 3.—Response of Structure (α = 10, β = 8, γ = 0) to Initial Velocity $\bar{\dot{x}}(0)$ = 2 for Two Conditions of Contact between Foundation Mat and Supporting Elements: (*a*) Bonded Contact Preventing Uplift; and (*b*) Unbonded Contact Only through Gravity with Uplift Permitted

from the displacements and velocities of the structure at the time when contact at both edges of the foundation mat is re-established.

Numerical results for the response of a structure to normalized initial velocity $\bar{\dot{x}}(0) = 2$ are presented in Fig. 3(*a*) for two conditions of contact between the foundation mat and the supporting spring-damper elements: (1) Bonded contact preventing uplift; and (2) unbonded contact only through gravity with uplift permitted. The response quantities presented as a function of time are deformation u, foundation-mat rotation θ, total lateral displacement $x = u + h\theta$, vertical displacements v_l and v_r of the left and right edges of the foundation mat, and uplift index which is zero if there is no uplift, +1 if the left edge uplifts and −1 if the right edge uplifts.

If the foundation mat is bonded to the supporting elements, the structural response is described by Eq. 9 for all time. The frequency of this simple harmonic motion is ω_1, defined in Eq. 5*a*, and the amplitudes of the response quantities are given by the critical values of Eq. 10 multiplied by the normalized initial velocity, which is 2 in this case.

The structural response is initially described by Eq. 9 even when uplift is permitted, resulting in response identical to the bonded case. When the deformation u and foundation-mat rotation θ reach their critical values (Eq. 10), the left edge of the mat uplifts at time a (Fig. 3(*a*)) and the subsequent response is described by Eq. 12. The deformation u oscillates with frequency λ_2 about a value $= u_c(h^2/h^2 + b^2)$ which for a wide range of α is close to u_c. This higher frequency response appears also in $v_l(t)$ and $v_r(t)$ and slightly in $\theta(t)$, but the total lateral displacement $x(t)$ varies smoothly with time without any noticeable contribution of the higher frequency mode. Contact at both edges of the foundation mat is re-established at time b and the response is described by Eq. 13 until the right edge uplifts at time c; at which time Eq. 12 takes over again until contact at both edges is re-established at time d and Eq. 13 is back in the picture. This response behavior is repeated during every cycle of free vibration as seen in Fig. 3(a). However, the response is almost, but not precisely, periodic because the initial conditions are not exactly repeated at the beginning of each vibration cycle. Whereas no vertical velocity is imparted to the structural mass when free vibration is initiated at $t = 0$, a small vertical velocity may be present at the beginning of subsequent cycles of free vibration. In addition to the slight differences in the response details from one cycle to the next, this perturbation can produce additional very short duration uplift episodes, as seen in the lowest portion of Fig. 3(a).

Because of uplift of the foundation mat, the maximum structural deformation u is reduced but the maximum values of all the other response quantities—θ, x, v_l and v_r—are increased. Noting that the time scale in Fig. 3(a) is normalized with respect to the natural vibration period T of the rigidly supported structure, it is apparent that the flexibility of the supporting elements has the effect of lengthening the vibration period to T_c, and uplift of the foundation mat results in even a longer period T_u.

Effects of Damping.—Expressed as a combination of modal contributions, the response of the undamped system to an initial velocity was given by Eq. 9 during the time that both edges of the foundation mat remain in contact with the supporting elements, by Eq. 12 during the time one edge is uplifted, and by Eq. 13 after contact has been re-established at both edges. When damping in the structure and the supporting elements is included, the responses of a system with massless foundation mat will be described by Eqs. 9, 12, and 13 with appropriated modifications to account for modal damping. In Eqs. 9 and 13, these include exponential decay terms $e^{-\xi_n\omega_n t}$ and $e^{-\xi_n\omega_n t'}$, respectively; and substitution of the damped frequency ω_n' instead of ω_n in the harmonic terms, where $\omega_n' = \omega_n\sqrt{1-\xi_n^2}$ and the modal damping ratios are

$$\xi_1 = \frac{\beta^3}{(\alpha^2+\beta^2)^{3/2}}\left(\xi + \frac{\alpha^2}{\beta^3}\xi_v\right) \qquad (14a)$$

$$\xi_2 = \xi_v \qquad (14b)$$

In Eq. 12 only the harmonic term is modified to include the exponential decay term $e^{-\zeta_2\lambda_2 t}$ and the damped frequency λ_2' is substituted instead of the undamped frequency λ_2, where $\lambda_2' = \lambda_2\sqrt{1-\zeta_2^2}$ and the damping ratio is

$$\zeta_2 = \frac{\beta^3(1+\alpha^2)^{1/2}}{(2\alpha^2+\beta^2)^{3/2}}\left(\xi + \frac{2\alpha^2}{\beta^3}\xi_v\right) \qquad (15)$$

For the structure with both edges of the foundation mat in contact with the supporting elements, the damping ratio for the second vibration mode—which involves uncoupled vertical motion without lateral deformation or base rotation—is simply ξ_v, the damping ratio of the structure in vertical vibration (Eq. 14b). In the first mode which includes lateral deformation of the structure coupled with rotation of the foundation mat about its c.g, the damping ratio is a linear combination of ξ_v and ξ, where the latter is the damping ratio of the rigidly supported structure in lateral vibration. For the linear system with uplift at one edge, the first vibration mode is a rigid-body mode without any damping. The damping ratio for the second vibration mode, which includes lateral structural deformation coupled with foundation-mat rotation, is a linear combination of ξ_v and ξ.

The response of the structure considered earlier, but now including damping in the structure as well as supporting elements, to normalized initial velocity $\bar{\dot{x}}(0) = 2$ is presented in Fig. 3(b). From the selected values $\xi = 0.05$, and $\xi_v = 0.4$, Eqs. 14 and 15 lead to $\xi_1 = 0.03$, $\xi_2 = 0.4$ and $\zeta_2 = 0.25$. Because of damping the tendency of the foundation mat to uplift, is reduced, resulting in the uplift duration decreasing with each vibration cycle; after a few cycles the foundation mat remains in a continuous contact with the supporting elements and the responses u, θ and x decay exponentially at a rate controlled by ξ_1. The vibratory term in the response during the time one edge is uplifted (Eq. 12) decays exponentially at a rate defined by ζ_2, which in this case is large and results in a quick decay of the high-frequency oscillations in u about the u_c value. The corresponding high-frequency effects in the other responses also disappear.

Based on the aforementioned considerations and numerical results of Fig. 3(b), we can examine the dependence of damping effects on the normalized initial velocity $\bar{\dot{x}}(0)$. If $\bar{\dot{x}}(0)$ exceeds 1 only slightly, the foundation mat will uplift from its supporting elements only for relatively short durations of time; for most of the time, neither edge of the foundation mat will uplift. The response of the structure will therefore be dominated by Eq. 13. The second mode does not contribute to the structural deformation, which will therefore decay exponentially at a rate controlled by the damping ratio ξ_1 for the first mode. As discussed earlier, this damping ratio is a linear combination of ξ and ξ_v.

If the normalized initial velocity $\bar{\dot{x}}(0)$ is much larger than 1, the foundation mat will remain in contact with its supporting elements only for a small fraction of a vibration cycle, too short to cause any significant decay of response. The left or right edge of the foundation mat will be uplifted during a major part of a vibration cycle. In this condition, the rotation and vertical displacement of the foundation mat are primarily due to the undamped rigid-body mode. The high-frequency ($= \lambda_2$) contributions to the structural deformation and vertical displacements at the edges of the foundation mat (Fig. 3(a)) essentially disappear due to damping (Fig. 3(b)), because they are due to the second mode (Eq. 12) which has a modal damping ζ_2. For typical parameter values λ_2 is several times higher than the overall frequency, thus several vibration cycles of the second mode occur during each uplift phase (Fig. 3(a)). The number of cycles is sufficient to essentially eliminate this high frequency response for a wide range of values of damping and other structural parameters. Considering that the rigid body mode is undamped and the high frequency response is eliminated by damping, the structural deformation response is insensitive to the damping values ξ and ξ_v if $\bar{\dot{x}}(0)$ is significantly larger than 1. This conclusion is supported by numerical results not presented here.

Analysis Procedure

The response of the system of Fig. 1(*a*) to specified ground motion can be analyzed by numerical solution of the equations of motion (Eq. 2). These equations are nonlinear as indicated by the dependence of the coefficients ϵ_1 and ϵ_2 on whether one or both edges of the foundation mat are in contact with the supporting elements. For each of the three contact conditions—contact at both edges, left edge uplifted, and right edge uplifted—the governing equations are linear, but they depend on the contact condition. Thus, during the time duration that one particular contact condition is valid, the system behaves as a linear system, but vibration properties of the linear system are different for the three contact conditions. Thus, at any time, the numerical analysis of response over the next time-increment consists of two steps: (1) Determining the contact condition valid for that time increment; and (2) solution of the equations of motion with the corresponding values for ϵ_1 and ϵ_2.

As mentioned earlier, for each contact condition the linear system has three degrees-of-freedom and the natural frequency of the third mode is at least an order of magnitude higher than the second mode frequency, and it tends to infinity as the mass of the foundation-mat approaches zero (see Appendix C, Ref. 10). Because the time-increment employed in numerical analysis of response is controlled by the vibration frequency of the highest mode, very short time increments must be used to accurately represent the contributions of the third mode. Because its frequency is at least an order of magnitude higher than that of the second mode, the time-increment would be so short to be almost impractical. Fortunately, the contribution of the third mode to the response of each of the three linear systems (corresponding to the three contact conditions) is negligible (see Appendix C, Ref. 10). By eliminating the effects of this high frequency mode in the numerical analysis, the time increment will be controlled by the natural frequencies of the first two vibration modes.

The obvious procedure to eliminate the high frequency effects is to express the structural displacements as a linear combination of the first two vibration modes of the system for the appropriate contact condition. However, this approach has the disadvantage that the choice of the time-varying modal coordinates depends on the contact condition. The more convenient approach is to transform the equations of motion to a set of displacements that can be used for all contact conditions. To this end the rotation θ is expressed in terms of u and v from Eq. 4:

$$h\theta = \frac{1}{\epsilon_1}\left[\frac{k}{k_f}\left(\frac{h}{b}\right)^2 u - \epsilon_2\left(\frac{h}{b}\right)v\right] \quad \ldots\ldots (16)$$

The displacement vector can therefore be expressed as

$$\begin{Bmatrix} u \\ h\theta \\ v \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{\epsilon_1}\dfrac{k}{k_f}\left(\dfrac{h}{b}\right)^2 & -\left(\dfrac{\epsilon_2}{\epsilon_1}\right)\dfrac{h}{b} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \ldots\ldots (17a)$$

$$\text{or} \quad \mathbf{r} = \mathbf{T}\bar{\mathbf{r}} \quad \ldots\ldots (17b)$$

Substituting the transformation of Eq. 17 into Eq. 2 and premultiplying both sides by the transpose of the transformation matrix **T** leads to a reduced system of two differential equations in the two unknowns u and v.

The eigenvalue problem associated with the reduced system of two differential equations was solved. The resulting natural frequencies and shapes of the two vibration modes were compared with the 'exact' values for the lower two modes, obtained from analysis of the original three-degree-of-freedom eigenvalue problem. The agreement was found to be excellent over a wide range of the system parameters γ, α and β (see Appendix C, Ref. 10). Thus, the reduced system of equations accurately retains the vibration properties of the lower two vibration modes, while eliminating the high-frequency third mode.

The reduced system of equations is integrated numerically using an implicit method with linear variation of acceleration in each time-step. Appropriate governing equations are used, consistent with the contact condition at the beginning of the time-step. The necessary modifications are incorporated if the contact condition changes during a time-step (see Appendix D, Ref. 10). By eliminating the high-frequency, third vibration mode it was possible to employ a much larger integration time-step than would otherwise be appropriate. The integration time-step $\Delta t = 0.01$ sec used in this investigation is ten times longer than that necessary to accurately solve the equations of motion in their original form (7), resulting in considerable saving in computational effort.

The procedure described to eliminate the contributions of the high-frequency, third vibration mode may be viewed as an application of the classical Rayleigh-Ritz method, wherein the two Ritz vectors are described by the columns of the transformation matrix **T**. If the foundation mat is massless, this procedure is equivalent to the static condensation approach outlined in a preceding section.

Earthquake Responses

The response of a structural system to the north-south component of the El Centro, 1940, ground motion computed by the numerical procedures described in the preceding section is presented in Fig. 4. Responses are shown for two conditions of contact between the foundation mat and the supporting spring-damper elements: (1) Bonded contact preventing uplift; and (2) unbonded contact only through gravity with uplift permitted. In the first case, the structural response is entirely due to the first natural vibration mode of the system with both edges of the foundation mat in contact with the supporting elements. Thus, the response behavior is similar to a single-degree-of-freedom system. When uplift of the foundation mat is permitted, the response behavior is much more complicated. During the initial phase of the ground shaking, both edges of the foundation mat remain in contact with the supporting elements. As the ground motion intensity builds up, the two edges of the foundation mat alternately uplift in a vibration cycle. In this example uplift occurs every vibration cycle during the strong phase of ground shaking, with the duration of uplift depending on the amplitudes of foundation mat rotation. As the intensity of ground motion decays to-

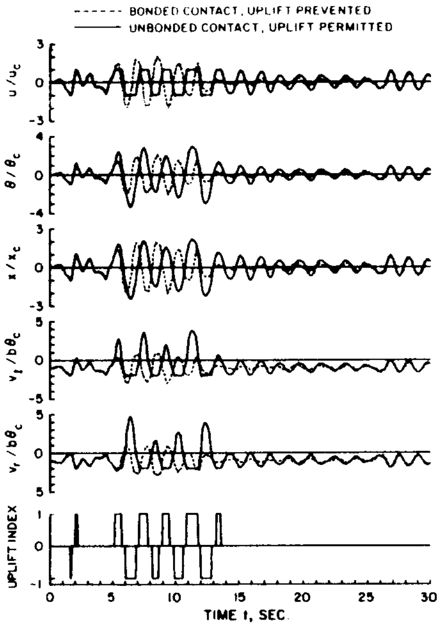

FIG. 4.—Response of Structure (α = 10, β = 8, γ = 0, T = 1 sec, ξ = 0.05, ξ_v = 0.4) to El Centro Ground Motion for Two Conditions of Contact between Foundation Mat and Supporting Elements: (*a*) Bonded Contact Preventing Uplift; and (*b*) Unbonded Contact Only through Gravity with Uplift Permitted

ward the later phase of the earthquake, the foundation mat no longer uplifts, and both edges remain in contact with the supporting elements.

The effects of foundation-mat uplift on the maximum response of the structure due to earthquake ground motion are similar to those observed in a preceding section during free vibration. The maximum deformation of the structure is close to u_c when foundation-mat uplift is permitted, which is a significant reduction compared to the response when foundation-mat uplift is prevented. Because of damping, the small-amplitude oscillations at the higher frequency λ_2 damp out and are not present in the earthquake response results of Fig. 4. The rotation of the foundation mat and vertical displacements of the two edges of the foundation mat are significantly increased due to the rigid body rocking mode of the system, permitted by uplift. This mode provides the dominant contribution to these responses during uplift but does not affect the structural deformation.

In order to study the effects of foundation-mat uplift on the maximum response of structures, response spectra are presented. The base shear coefficient

$$V_{\max} = \frac{\mathbf{V}_{\max}}{w} = \frac{ku_{\max}}{mg} = \left(\frac{2\pi}{T}\right)^2 \frac{u_{\max}}{g} \quad \text{(18)}$$

in which $\mathbf{V}_{\max}$ = the maximum base shear, and w = the weight of the

superstructure, is plotted as a function of the natural vibration period of the corresponding rigidly supported structure. For each set of system parameters α, β, γ, ξ and ξ_v, such a response spectrum plot is presented for two conditions of contact between the foundation mat and the supporting spring-damper elements: (1) Bonded contact preventing uplift; and (2) unbonded contact only through gravity with uplift permitted. Also presented are the results for the corresponding rigidly supported structure, which is simply the standard pseudoacceleration response spectrum, normalized with respect to gravitational acceleration. Included in the response spectra plots is V_c, the critical base shear coefficient associated with the maximum value of base shear $\mathbf{V}_c$ obtained from static considerations (Eq. 1*a*):

$$V_c = \frac{\mathbf{V}_c}{w} = \frac{1+\gamma}{\alpha} \quad \text{(19)}$$

This critical base shear coefficient depends on the mass ratio γ and slenderness-ratio parameter α, but is independent of the vibration period T.

The response spectra presented in Fig. 5 are for systems with massless foundations (γ = 0) and a fixed set of system parameters α, β, ξ, and ξ_v subjected to the El Centro ground motion. The differences between the response spectra for the two linear systems, the structure with foundation mat bonded to the supporting elements and the corresponding rigidly supported structure, are due to the change in period and damp-

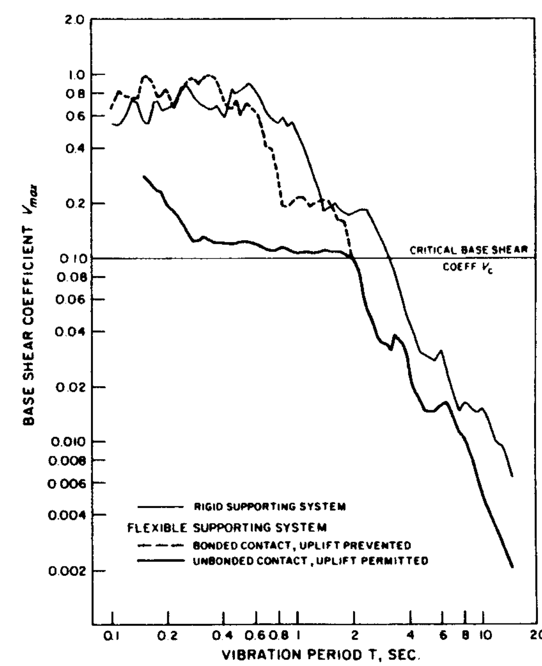

FIG. 5.—Response Spectra for Structures (α = 10, β = 8, γ = 0, ξ = 0.05, ξ_v = 0.4) Subjected to El Centro Ground Motion for Three Support Conditions

ing resulting from support flexibility (Eqs. 5 and 14). The base shear developed in structures with relatively long vibration periods is below the critical value and the foundation mat does not uplift from its supporting elements. If foundation mat uplift is prevented, the maximum base shear at some vibration periods may exceed the critical values; for the selected system parameters and ground motion Fig. 5 indicates that the critical value is exceeded for all vibration periods shorter than the period where the linear spectrum first attains the critical value. If the foundation mat of such a structure rests on the spring-damper elements only through gravity and is not bonded to these elements, uplift occurs and this has the effect of reducing the base shear. However, the base shear is not reduced to as low as the critical value based on static considerations (Eq. 1*a*). Although foundation-mat uplift is initiated at the time instant the base shear in a vibrating structure reaches the instantaneous dynamic critical value, depending on the state—displacement, velocity and acceleration—of the system the deformation and base shear may continue to exceed the critical values. Furthermore the dynamic, critical base shear may exceed the static critical value because the time-dependent vertical force p may exceed the gravity loads. Because the base shear developed in linear structures (foundation-mat uplift prevented) tends to exceed the critical value by increasing margins as the vibration period decreases, the foundation mat of a shorter-period structure has a greater tendency to uplift, which in turn causes the critical base shear to be exceeded by a greater margin, although it remains well below the linear response.

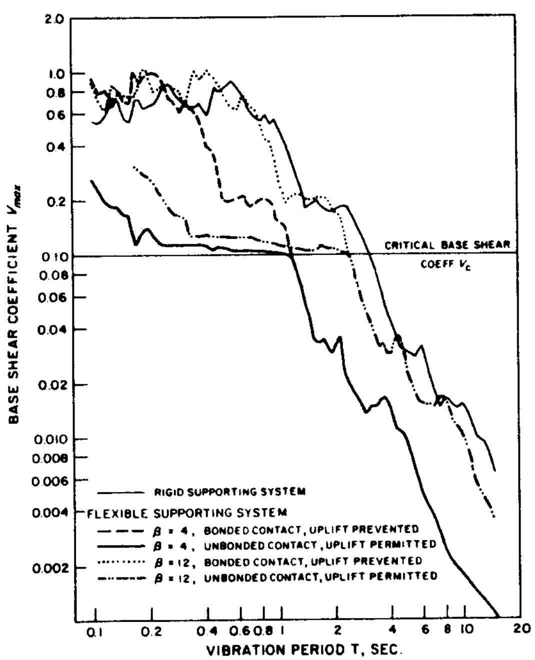

FIG. 6.—Response Spectra for Structures (α = 10, γ = 0, ξ = 0.05, ξ_v = 0.4) Subjected to El Centro Ground Motion for Two Values of Frequency Ratio β = 4 and 12

The response spectra presented in Fig. 6 are for two values of the frequency ratio β with all other system parameters kept constant. As mentioned earlier, the differences in the response spectra for the two linear systems, the structure with foundation mat bonded to the supporting elements and the corresponding rigidly supported structure, are due to the change in period and damping resulting from support flexibility (Eqs. 5 and 14). The period change appears as a shift in the response spectrum to the left, with larger shift for smaller values of β, i.e., for the more flexible supporting elements. The nonlinear response spectrum for the structure with foundation mat permitted to uplift shifts similarly to the left, with uplift initiated at longer periods as β increases. But for the period shift, the shape of the linear as well as nonlinear response are affected little by the frequency ratio β.

As presented in Eq. 19, the critical base shear coefficient is inversely proportional to the slenderness-ratio parameter α. This would suggest that maximum base shear will be smaller for relatively slender structures, which is confirmed by response spectra presented in Fig. 7. The foundation mat of a slender structure has a greater tendency to uplift resulting in greater reductions in the base shear. Uplift of the foundation

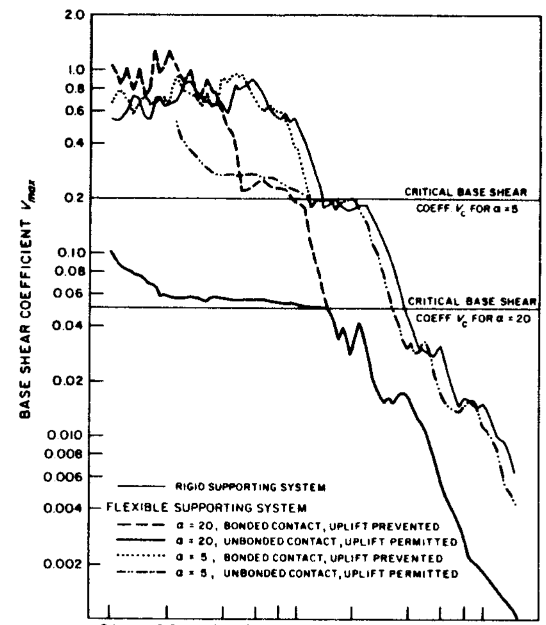

FIG. 7.—Response Spectra for Structures (β = 8, γ = 0, ξ = 0.05, ξ_v = 0.4) Subjected to El Centro Ground Motion for Two Values of Slenderness Ratio Parameter α = 5 and 20

mat occurs at all vibration periods shorter than the period where the linear response spectrum attains the critical value. This period depends on the slenderness ratio α in a complicated manner, because the critical base shear coefficient as well as the period shift of the linear response spectrum (relative to the standard pseudo-acceleration spectrum) both depend on α. For relatively slender structures the critical base shear coefficient is smaller (Eq. 19), but the period shift is larger (Eq. 5).

Summary and Conclusions

The equations of motion for the structure with its foundation mat permitted to uplift are nonlinear with foundation stiffness and damping parameters dependent on whether the foundation mat is in contact with the supporting systems at one or both edges. However a linear system corresponding to any instantaneous contact condition of the foundation mat can be defined. This linear system has three degrees-of-freedom and the natural frequency of the third mode is at least an order of magnitude higher than the second mode frequency and it tends to infinity as the foundation-mat mass approaches zero. Because the contribution of this high frequency mode to the response is negligible, the Rayleigh-Ritz concept was employed to eliminate this mode from the analysis. It was then possible to employ a much larger integration time-step than would otherwise be practical, resulting in considerable savings in computational effort.

Analytical examination of the governing equations and system properties and the numerical results, presented here and elsewhere (10), indicate that the earthquake response of uplifting structures is controlled by the following system parameters, listed in more or less descending order of importance: (1) Natural vibration frequency ω of the rigidly supported structure; (2) slenderness-ratio parameter α; (3) ratio γ of foundation mass to superstructure mass; (4) $\beta = \omega_v/\omega$ where ω_v is the vertical vibration frequency of the system with its foundation mat bonded to supporting elements; (5) damping ratio ξ of the rigidly supported structure; and (6) damping ratio ξ_v in vertical vibration of the system with its foundation mat bonded to the supporting elements.

If we had considered the large-amplitude motion of the structure including the possibility of its overturning the response of a structure would be influenced also by its size. Similarly the size influences even the small-amplitude response if p-δ effects are considered, but the results presented indicate that these effects are insignificant for most buildings (10).

A study of the response spectra plots presented leads to the following conclusions:

1. The base shear developed in structures with relatively long vibration periods is below the critical value, the maximum base shear that can be developed with the structure supported only through gravity, and the foundation mat does not uplift from its supporting elements.
2. For short period structures, the base shear exceeds the critical value if foundation-mat uplift is prevented; for such structures, permitting uplift has the effect of reducing the base shear—to values somewhat above the critical value.

3. The foundation mat of a slender structure has a greater tendency to uplift resulting in greater reductions in base shear.
4. The shape of the response spectrum is affected little by the frequency ratio β, but it does have the effect of shifting the response spectrum to the left with larger shift for smaller β, i.e., for the more flexible supporting elements.

The possibility of transient uplift of a portion of the foundation mat or of a few individual footings, as the case may be, should be considered in the analysis of response of structures subjected to intense earthquake ground motion. Because foundation uplift generally has the effect of reducing the structural deformations and forces, there is no need to prevent it but, on the contrary, it is desirable to permit it. The foundation, underlying soil and the structural columns should be properly designed to accommodate the transient uplift and the effects of subsequent impact on contact.

Although the maximum base shear is reduced because of foundation-mat uplift for slender structures over a wide range of fixed-base vibration period, very short period structures may be exceptions to this behavior. Their response, as seen in results presented elsewhere (1), may increase because of foundation mat uplift. However these vibration periods are unrealistically short for slender structures representative of medium-rise to high-rise structures with longer vibration periods. Thus, the increased computed response of very short period structures in spite of foundation mat uplift is of little practical consequence.

Because they are based on the response results presented here and in the more complete report (10), all for one selected ground motion, strictly speaking the aforementioned conclusions may not be valid for other excitations or range of structural parameters. However, as observed elsewhere (1), the response behavior of structures, excluding those with very short fixed-base vibration periods, is simple when their foundation mat is permitted to uplift even though the system is nonlinear. In a sense the behavior seems to be even simpler than that of the corresponding linear system—the structure with its foundation mat bonded to the supporting soil. In particular, the maximum base shear of uplifting structures does not depend significantly on the details or intensity of the ground acceleration $\ddot{u}_g(t)$, but is controlled by the critical shear V_c (Eq. 1*a*) for the structure (10). Thus it seems that the aforementioned conclusions should be also valid for a variety of ground motions.

Although the results of this work have provided an understanding of the basic effects of transient foundation uplift on the earthquake response of structures, there is an important limitation in the application of these results to actual buildings. In reality the foundation stiffness and damping parameters depend on the displacement amplitude and, excitation frequency, in contrast to the constant parameters employed in this work. In either case, the foundation parameters are very difficult to evaluate because they would depend on the details of the foundation design, including the degree of foundation embedment and the base deformability. Reliable methods to evaluate foundation parameters for actual buildings need to be developed, so that the beneficial effects of foundation uplift can be considered in the design of buildings.

Acknowledgment

This research was conducted under Grants PFR79-08261 and CEE81-05790 from the National Science Foundation for which the authors are grateful.

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Appendix II.—Notation

The following symbols are used in this paper:

b = half width of foundation mat;
c = lateral damping coefficient of superstructure;
c_f = damping coefficient of supporting element of two-element foundation;
g = acceleration of gravity;
h = height of superstructure;
k = lateral stiffness of superstructure;
k_f = stiffness of supporting element of two-element foundation;
M = static base moment applied at c.g of foundation mat;

M_c = maximum static base moment at full uplift of foundation mat with contact reduced to one edge;
m = mass of superstructure;
m_o = mass of foundation mat;
R_l = upward reaction of left supporting element of two-element foundation;
R_r = upward reaction of right supporting element of two-element foundation;
T = natural vibration period of the rigidly supported structure;
T_c = rocking vibration period of flexibly supported structure with uplift prevented;
T_u = rocking vibration period of flexibly supported structure with uplift permitted;
u = structural deformation;
u_c = maximum structural deformation under the action of static lateral force;
$\ddot{u}_g(t)$ = horizontal ground acceleration;
u_{max} = maximum structural deformation during an earthquake;
V_{max} = a base shear coefficient = $\mathbf{V}_{max}/w$;
V_c = a base shear coefficient = $\mathbf{V}_c/w$;
$\mathbf{V}_c$ = maximum base shear that can be developed under the action of static lateral force;
$\mathbf{V}_{max}$ = maximum base shear;
v = vertical displacement of c.g of foundation mat;
v_l = vertical displacement of left edge of foundation mat;
v_r = vertical displacement of right edge of foundation mat;
v_s = vertical displacement of foundation mat under the action of gravity forces;
w = weight of superstructure;
x = lateral displacement of structure relative to supporting foundation = $u + h\theta$;
$\dot{x}(0)$ = initial velocity of structure;
$\dot{x}(0)_c$ = minimum initial velocity of structure inducing foundation-mat uplift of structure supported on two-element foundation;
$\bar{\dot{x}}(0)$ = normalized initial velocity of structure;
α = slenderness ratio parameter;
β = ω_v/ω;
γ = ratio of foundation mass to superstructure mass;
ζ_i = damping ratio of the ith vibration mode after uplift;
θ = rotation of foundation mat;
θ_c = static rotation of foundation mat at full uplift with contact reduced to one edge;
λ_i = undamped natural vibration frequency of the ith mode after uplift;
ξ = damping ratio of the rigidly supported structure;
ξ_v = damping ratio in vertical vibration of the system of Fig. 1 with its foundation mat bonded to the two-element foundation;
ξ_i = damping ratio of the ith vibration mode of system with both edges of foundation mat in contact with foundation;
ϕ_i = ith vibration mode of system with foundation mat in contact with both edges of foundation;

ψ_i = ith vibration mode of system after uplift;
ω = natural vibration frequency of the rigidly supported structure;
ω_i = natural vibration frequency of the ith mode of system with both edges of foundation mat in contact with supporting elements; and
ω_v = vertical vibration frequency of the system with its foundation mat bonded to the supporting elements.

On Free Liquid Oscillations in Irregular Basins

By Helmy M. Safwat[1]

Abstract: Complex variable methods have been devised to investigate the effect of geometric irregularities on the dynamic characteristics of inviscid liquids in circular basins. The fundamental frequency of the slosh mode has been determined by the stationary property of Schwarz quotients. Lower and upper bounds for the frequency have been obtained by a comparison theorem. Results show that frequencies are decreased with the increase of out of roundness parameters.

Introduction

Under the influence of the gravitational field, a liquid with a free surface, lying in a container, can oscillate. One of the important examples of this type of liquid motion is the oscillations in lakes, harbors with almost closed entrances and closed basins (11). The determination of the natural oscillation of liquids in these systems is related to the problem known as "Harbor Resonance." Exact solutions exist for some simple shaped basins such as rectangular and circular basins (16), and recently (8,9) for liquid sheets in an eccentric-annulus cross section. For other shapes, naturally, one must resort to approximate methods e.g., finite differences (13,15), finite elements (4,14) and conformal mapping—Galerkin method (6,7).

The present paper deals with the study of the effect of geometric irregularities and deviations from the ideal circular shape on the fundamental frequency of natural oscillations of a liquid filling to an arbitrary depth such as a basin whose planform has such an irregular shape. The fundamental frequency has been determined by a systematic application of the two complex variables method in the unit disk plane to a quotient known as "Schwarz Quotient."

Fundamental Equations

Consider a basin or a harbor with an almost closed entrance that has a horizontal boundary, ∂D filled with water up to height h. Taking the origin, O, at any point of the undisturbed free surface of the liquid and setting a right handed orthogonal triad OXYZ with the axis OZ pointed vertically upward, the oscillatory motion of the inviscid liquid can be described by a single-valued potential function $\Phi(X, Y, Z; t)$ that satisfy (5):

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Y^2} + \frac{\partial^2\Phi}{\partial Z^2} = 0 \quad (Z \leq 0) \qquad (1)$$

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Note.—Discussion open until December 1, 1984. To extend the closing date one month, a written request must be filed with the ASCE Manager of Technical and Professional Publications. The manuscript for this paper was submitted for review and possible publication on September 28, 1983. This paper is part of the *Journal of Engineering Mechanics*, Vol. 110, No. 7, July, 1984. ©ASCE, ISSN 0733-9399/84/0007-1147/$01.00. Paper No. 18984.