Nonlinear Impact and Chaotic Response of Slender Rocking Objects

By Solomon C. S. Yim,[1] Associate Member, ASCE, and Huan Lin[2]

Abstract: This investigation focuses on the influence of nonlinearities associated with impact on the behavior of the rocking response of free-standing rigid objects subjected to horizontal base excitations. The object is to identify the causes that make the rocking response difficult to predict and that have made past experiments with identical setups and excitations unrepeatable. The impact nonlinearities examined are: (1) The transition of governing equations; and (2) the abrupt reduction in angular velocity associated with impact at the base. In addition to the periodic and overturning responses, the existence of two new types of (bounded) response not previously revealed in literature—quasi-periodic and chaotic are discovered in this study. An approximate method based on the Melnikov function to analytically predict the existence of chaotic response is derived. Modern geometric and numerical identification techniques are employed. The accuracy of the method is assessed by numerical results. The relationship between the Melnikov analysis developed in this study and the stability analysis developed by Spanos and Koh is examined. It is shown that the divergent nature of the rocking motion in conjunction with the transition nonlinearity associated with impact constitute the major cause of the extreme sensitivity of the rocking response.

Introduction

The rocking behavior of rigid objects has been of interest to civil engineers for over a century. Many structures, from ancient historical minarets, monumental columns, and tombstones, to modern-day petroleum storage tanks, water towers, nuclear reactors, and concrete radiation shields, can be considered free-standing objects. They may be subjected to support excitations due to earthquake ground motion, and/or nearby machine vibrations. An understanding of the rocking behavior of these objects is of vital importance for safe operations, preservation of existing structures, and the design of new equipment.

Recent studies have demonstrated that the rocking response of rigid objects can be very sensitive to the system parameters and the ground-motion details. In a series of experimental dynamic tests performed on the Berkeley shaking table, Aslam et al. (1978) showed that the rocking response was so sensitive that some of the experiments were deemed nonrepeatable. Using a single-degree-of-freedom (SDOF) model for the system, Yim et al. (1980) examined the sensitivity of the rocking response by conducting a numerical study to identify, in a statistical sense, the parametric dependency of overturning stability on the size, slenderness ratio, coefficient of restitution, and ground-motion intensity. The deterministic results confirmed that the rocking response can be sensitive to small changes in system parameters, and probabilistic trends can only be established with a large sample size.

Recognizing the insurmountable difficulties in the analysis of the complex behavior of the fully nonlinear rocking objects subjected to earthquake excitations, Spanos and Koh (1984) and Tso and Wong (1989) simplified the

[1]Assoc. Prof., Dept. of Civ. Engrg., Oregon State Univ., Corvallis, OR 97331.
[2]Grad. Res. Asst., Dept. of Civ. Engrg., Oregon State Univ., Corvallis, OR 97331.

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SDOF system by assuming the rigid objects to be slender and the base excitation to be harmonic, thus allowing linearization of the individual governing equations of motion and removing the randomness in the excitation. They were able to develop approximate analytical methods to predict the existence and stability of harmonic and subharmonic responses. As a result of these studies, significant advances in the understanding of rocking behavior were made. However, as Wong and Tso (1989) pointed out in their experimental study of the simplified system, there appear to be responses that cannot be accounted for by the analytical methods developed to date.

This investigation focuses on the influence of nonlinearities associated with impact on the behavior of the rocking response of a simplified system using modern analytical and numerical techniques. The objective is to clearly identify the causes that make the rocking response difficult to predict and that make experiments with identical setups and excitations unrepeatable. The nonlinearities examined are: (1) The transition of governing equations when the center of rotation changes from one edge to the other; and (2) the abrupt reduction in angular velocity and the associated energy dissipation.

In addition to the periodic and overturning responses, the existence of two new types of bounded responses not previously revealed in literature, quasi-periodic, and chaotic, are discovered in this study. Geometric methods are developed to identify the responses and to characterize the behavior of the response. An analytical method based on the Melnikov function to predict the existence of chaotic response is derived. The accuracy of the method is assessed by numerical results. The relationship between the Melnikov analysis method and the stability analysis method developed by Spanos and Koh (1984) is examined.

Systems Considered

The free-standing slender object is modeled as a rectangular rigid body subjected to horizontal base motion excitation (Fig. 1). Assuming that the coefficient of friction is sufficiently large so that there will be no sliding between the object and the base, depending on the support accelerations, the object may move rigidly with the base or be set rocking. If rocking occurs, it is assumed that the body will oscillate rigidly about the centers of rotation O and O'. The governing equation of motion for the rigid object with positive angular rotation about corner O is as follows [see Yim et al. (1980)]:

$$I_o\ddot{\theta} + MRa_{gx} + MgR(\theta_{cr} - \theta) = 0, \qquad \theta > 0 \qquad (1a)$$

where H and B = the height and width of the object; I_o = moment of inertia about O; M = mass; a_{gx} = the horizontal base acceleration; R = the distance from O to the center of mass; and $\theta_{cr} = \cot^{-1}(H/B)$ = the critical angle beyond which overturning will occur for the object under gravity alone. Similarly, the rocking about O' is governed by the equation

$$I_o\ddot{\theta} + MRa_{gx} - MgR(\theta_{cr} + \theta) = 0, \qquad \theta < 0 \qquad (1b)$$

Impact occurs when the angular rotation crosses zero, approaching from either a positive or negative direction, and the base surfaces recontact. Associated with the impact is a transition from rocking about one corner to rocking about the other and a finite amount of energy loss (or damping), which can be accounted for by reducing the angular velocity of the object after impact. As in Yim et al. (1980), it is assumed that the angular velocity before and after impact is related by a parameter e through the following equation:

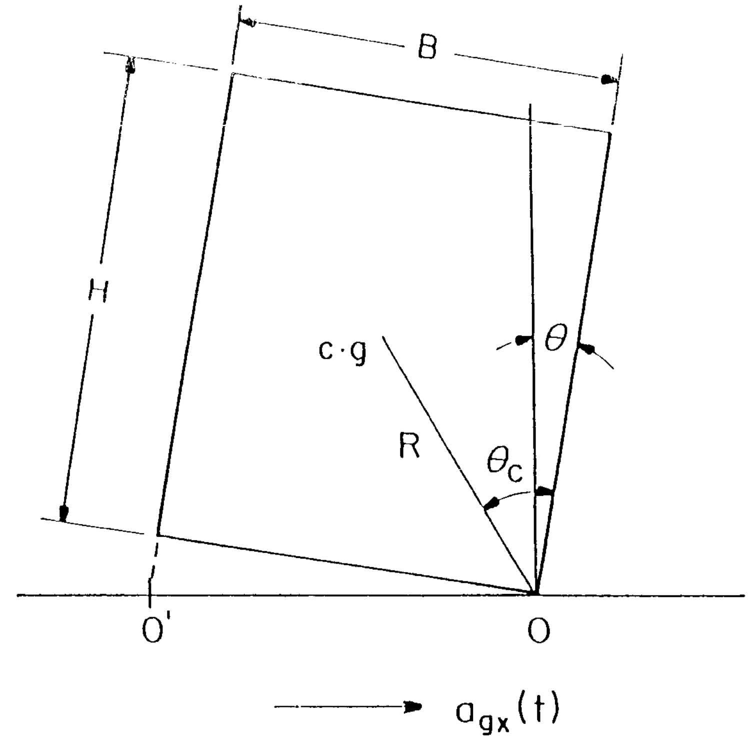

FIG. 1. Idealization of Free-Standing Equipment as Rigid Rocking Object to Horizontal Excitation

$$\dot{\theta}(t^+) = e\dot{\theta}(t^-), \qquad 0 \le e \le 1 \qquad (2)$$

where e = the coefficient of restitution; t^+ = the time just after impact; and t^- = the time just before impact. In this study, the horizontal base excitation is assumed to be harmonic with constant amplitude and a single frequency

$$a_{gx} = a_x \cos(\omega t + \phi) \qquad (3)$$

Note that there are two nonlinearities in the system associated with impact. The first results from the transition from one governing equation to the other as the center of rotation changes from one edge to the other. The second is due to the abrupt reduction (jump discontinuity) in the angular velocity caused by the impact energy dissipation (damping). In the limiting case, the damping nonlinearity can be removed if it is assumed that the material of the rigid object and the base support are stiff and that the rebound following impact is perfectly elastic so that there is no energy loss, i.e., $e = 1$.

Method of Analysis

Because the individual equations governing the rocking motion about each edge ($1a$) and ($1b$) are linear, exact analytical solutions for these equations exist and are used in conjunction with the nonlinear transition conditions at impact in determining the responses (Spanos and Koh 1984). By letting τ

frequency, respectively, the piece-wise linear versions of (1a) and (1b) can be reduced to

$$\ddot{\Theta} - \Theta = -A_x \cos(\Omega\tau + \phi) - 1, \qquad \Theta > 0 \quad (4a)$$

$$\ddot{\Theta} - \Theta = -A_x \cos(\Omega\tau + \phi) + 1, \qquad \Theta < 0 \quad (4b)$$

The solutions of (4a) and (4b) are

$$\Theta^+(\tau) = a^+ \sinh\tau + b^+ \cosh\tau + 1 + \beta\cos(\Omega\tau + \phi) \quad (5a)$$

$$\Theta^-(\tau) = a^- \sinh\tau + b^- \cosh\tau - 1 + \beta\cos(\Omega\tau + \phi) \quad (5b)$$

The superscript (+) = it is valid for $\Theta > 0$; and the superscript (−) = it is valid for $\Theta < 0$. The symbols a^+, b^+, a^- and b^- = constants of integration dependent on the initial conditions, and $\beta = A_x/(1 + \Omega^2)$. The response to harmonic excitations can be obtained by selecting the proper equation with the appropriate initial conditions and increasing the time variable successively. If at a particular time the magnitude of Θ exceeds 1.0 and continues to diverge, the rigid object will overturn. If Θ returns to zero, then the corresponding angular velocity can be determined, and the solution given by the other equation can be applied. This procedure is then repeated until steady-state response is obtained.

Classification of Responses

When subjected to harmonic horizontal base excitation, depending on the amplitude and frequency of the excitation, the response of a slender object may either be unbounded, which leads to overturning, or it will eventually settle into a bounded motion. In the past, it was thought that, under periodic excitations, all bounded responses were periodic. Thus, the study of bounded responses concentrated on periodic motions (Spanos and Koh 1984). In the following sections, it will be shown that there exist two additional types of bounded responses, namely quasi-periodic, and chaotic responses. The formal definitions of these responses are introduced, and their time histories and phase diagrams are presented. For completeness, periodic and overturning responses are also briefly reviewed.

Periodic Response

A steady-state bounded response of a dynamical system is periodic if the motion repeats itself with a fixed time T, i.e.

$$x(t + T) = x(t) \quad (6)$$

for all time t, where T = the period of the motion. It is well known that for nonlinear systems, in addition to primary resonance response, subharmonic, superharmonic, and super-subharmonic responses are also possible. For the range of excitation parameters of the rocking system considered in this study, harmonic and subharmonic responses are found to occur frequently. The time history and phase diagrams of typical examples of a harmonic response, a one-third subharmonic response, and a one-tenth subharmonic response of the rocking system are shown in Figs. 2 and 3, respectively.

Overturning Response

For some combinations of excitation amplitude and frequency, the magnitudes of the angular rotation response, after a finite time, do not return to zero but increase without bound with time. Physically, this type of response

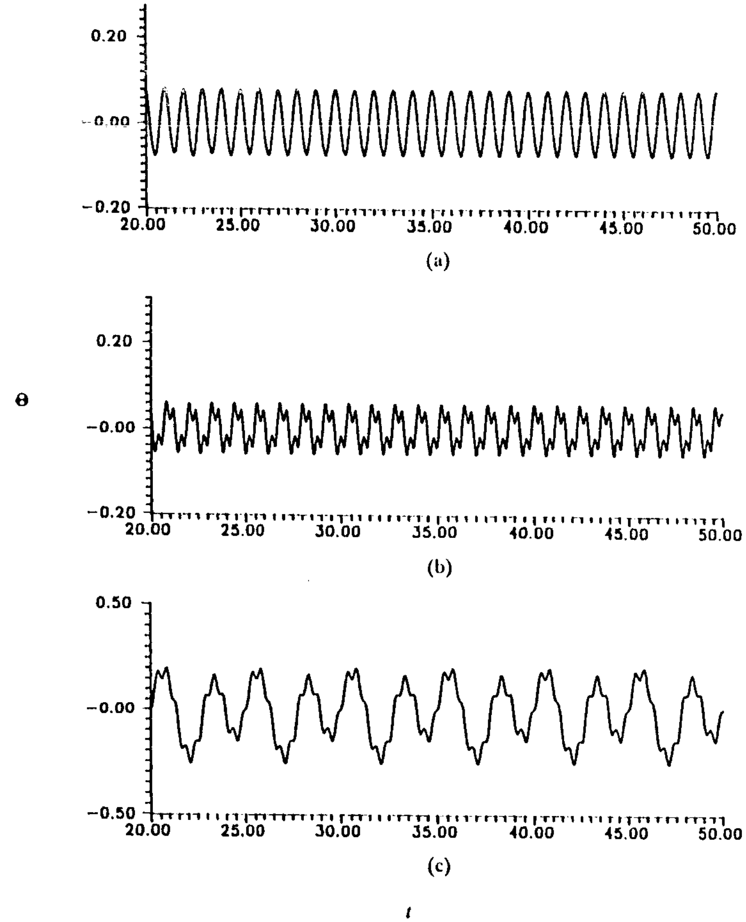

FIG. 2. Time Histories of Periodic Response: (*a*) (1,1) Mode, A_x = 2.0, T = 1.0; (*b*) (1,3) Mode, A_x = 6.5, T = 0.4; (*c*) Unsymmetric (1,10) Mode, A_x = 5.5, T = 0.5, $[(\dot{\Theta}, \phi_*)$ = (0.15,1.8)], e = 0.925

leads to overturning when the magnitude of the angular rotation far exceeds the critical angle. Typically, for fixed-system parameters, overturning response is associated with large excitation amplitudes. The time history and phase diagram of a typical example of an overturning response of the rocking system are shown in Fig. 4.

Quasi-Periodic Response

A new type of bounded rocking response discovered in this study is the quasi-periodic motion. It is a combination oscillation consisting of two or more incommensurate freuencies. The simplest example of combination oscillation takes the form

$$x(t) = b_1 \cos\omega_1 t + b_2 \cos\omega_2 t \quad (7)$$

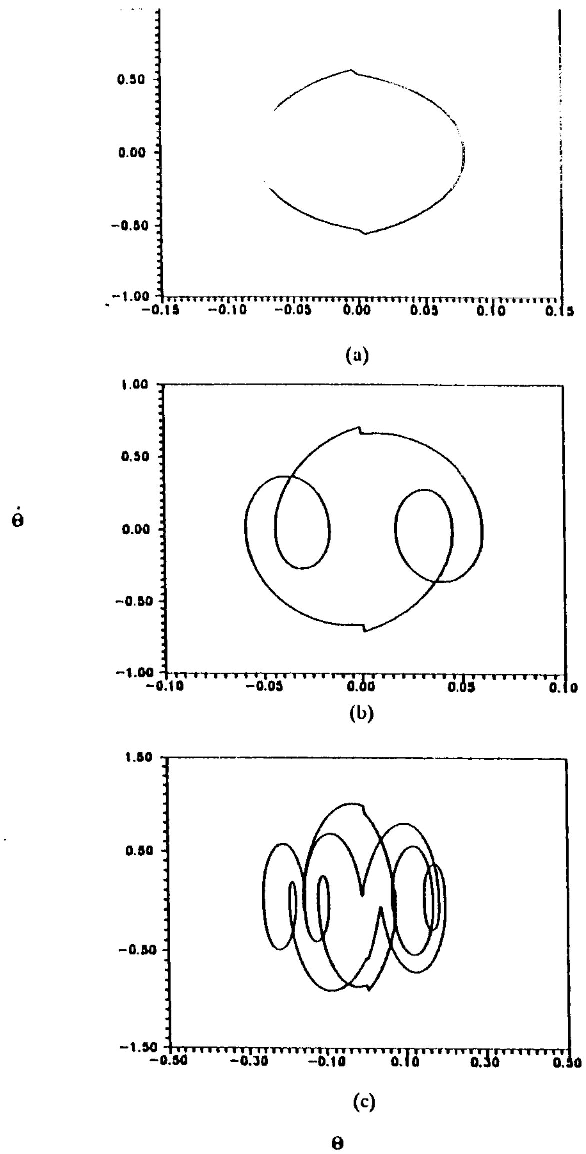

FIG. 3. Phase Diagrams of Periodic Response: (*a*) (1,1) Mode, A_x = 2.0, T = 1.0; (*b*) (1,3) Mode, A_x = 6.5, T = 0.4; (*c*) Unsymmetric (1,10) Mode, A_x = 5.5, T = 0.5, $[(\dot{\Theta}, \phi_s) = (0.15, 1.8)]$, e = 0.925

where ω_1 and ω_2 are incommensurate, i.e., ω_1/ω_2 is an irrational number. In this case, a periodic motion is modulated in some way by a second motion, which is periodic but with different period [for a general description of quasi-periodic response, see Moon (1987)]. The time history and phase

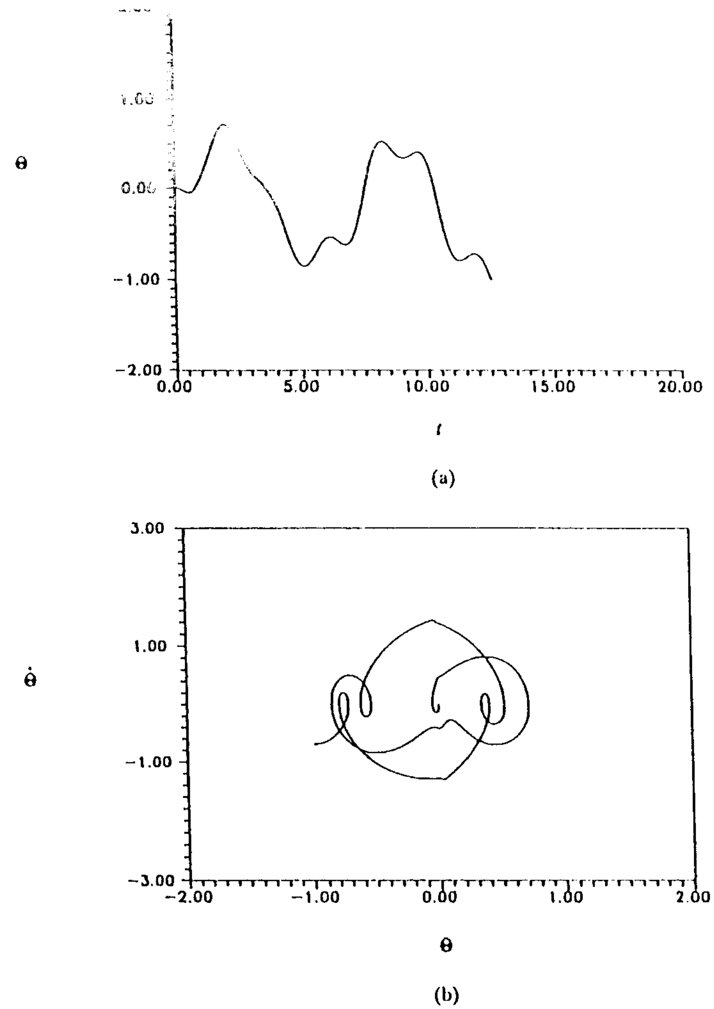

FIG. 4. Overturning Response: (*a*) Time History; (*b*) Phase Diagram, A_x = 1.75, T = 2.0, and e = 0.95

diagram of a typical example of a quasi-periodic response of the rocking system are shown in Fig. 5.

Chaotic Response

A second new type of bounded rocking response discovered in this study is chaotic motion. Its response behavior is characterized by a random-like, unpredictable aspect as well as a certain order in the motion, although the excitation is straightly deterministic and periodic. The unpredictability arises from its extreme sensitive dependence on initial conditions. Nearby trajectories of chaotic motion, with seemingly infinitesimally different initial conditions, will diverge exponentially, leading to large differences in long-term

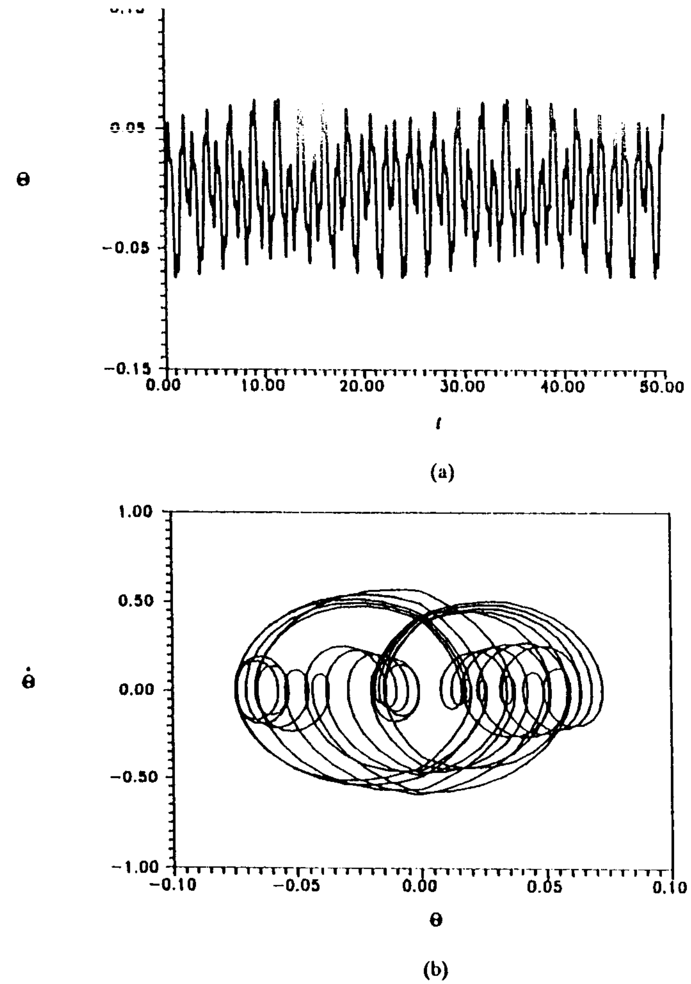

FIG. 5. Quasi-Periodic Response: (*a*) Time History; (*b*) Phase Diagram, A_x = 4.0, T = 0.4, and e = 1.0

predictions of the responses [for a generarl description of chaotic response, see Thompson and Stewart (1986)]. The time history and phase diagram of a typical sample of chaotic response of the rocking system are shown in Fig. 6.

RESPONSE IDENTIFICATION

As shown in the preceding sections, periodic (bounded) and overturning (unbounded) responses of the rocking object system can be readily identified by inspection of their time histories and phase diagrams. However, in general, the difference between quasi-periodic and chaotic responses is not always obvious from the time histories and phase diagrams. To identify and

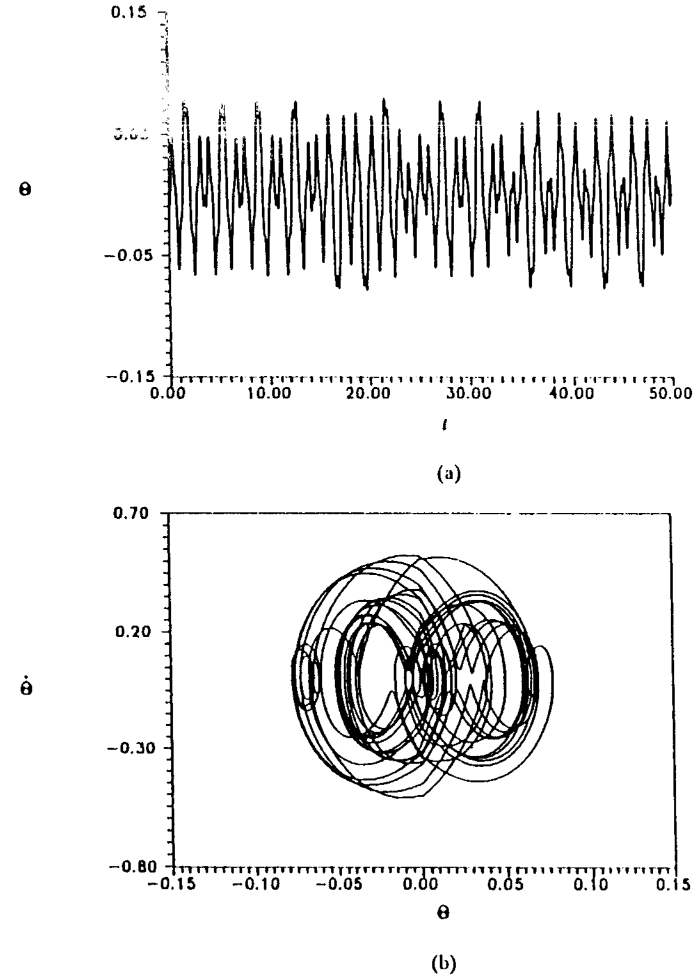

FIG. 6. Chaotic Response: (*a*) Time History; (*b*) Phase Diagram, A_x = 2.0, T = 0.4, and e = 1.0

differentiate the two types of (bounded) responses, the Poincaré map, the spectral density, the Lyapunov exponent, and the fractal dimension are employed in this study. Readers interested in a detailed technical discussion of these identification techniques may see Moon (1986) and Thompson and Stewart (1987).

The Poincaré maps of the one-third subharmonic, quasi-periodic, and chaotic responses of the rocking systems presented in the preceding sections are shown in Fig. 7. Note that the map for the one-third subharmonic response contains three fixed points, the quasi-periodic response forms three tori, and the chaotic response generates a pattern called a strange attractor.

The spectral density functions of the given example responses are shown in Fig. 8. The periodic, quasi-periodic, and chaotic responses have a sin-

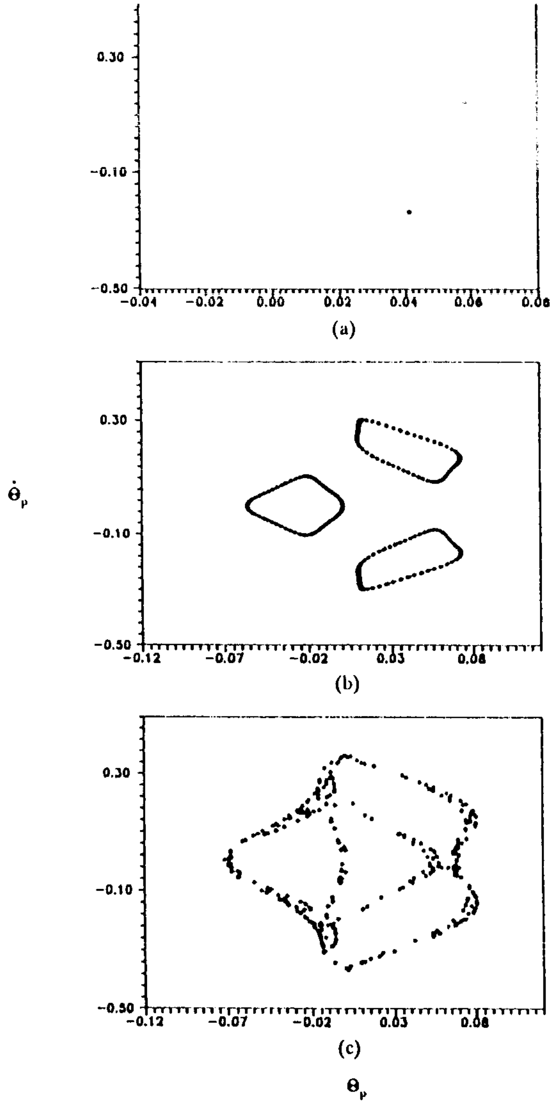

FIG. 7. **Poincaré Maps: (*a*) Subharmonic Response, $A_x = 6.5$, $T = 0.4$, and $e = 0.925$; (*b*) Quasi-Periodic Response, $A_x = 4.0$, $T = 0.4$, and $e = 1.0$; (*c*) Chaotic Response, $A_x = 2.0$, $T = 0.4$, and $e = 1.0$**

gle dominant frequency, a finite number of incommensurate frequencies, and an infinite number of frequencies (i.e., a broadband spectrum), respectively.

The Lyapunov exponent is a quantitative measure of the divergence rate of nearby trajectories in phase space. The Lyapunov exponent of chaotic response is positive, whereas those of periodic and quasi-periodic responses

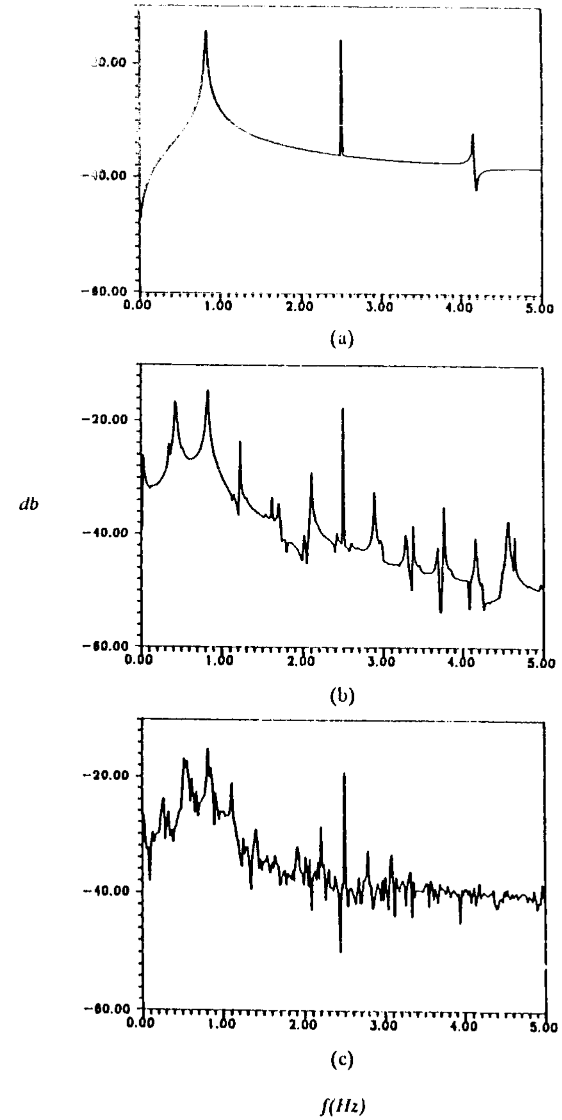

FIG. 8. **Spectral Densities: (*a*) Subharmonic Response, $A_x = 6.5$, $T = 0.4$, and $e = 0.925$; (*b*) Quasi-Periodic Response, $A_x = 4.0$, $T = 0.4$, and $e = 1.0$; (*c*) Chaotic Response, $A_x = 2.0$, $T = 0.4$, and $e = 1.0$**

are zero. For the responses presented, the computed Lyapunov exponent for the periodic, quasi-periodic, and chaotic responses are indeed 0, 0, and 0.14, respectively.

A fractal dimension is a quantitative property of a set of points in an n-dimensional space that measures the extent to which the points fill a subspace. A noninteger dimension is a hallmark of chaotic response. The fractal dimension of the periodic, quasi-periodic, and chaotic responses presented are 0, 2.0, and 2.3, respectively.

Existence and Stability of Periodic Response

Conditions for the existence and stability of periodic responses of slender rocking objects to harmonic horizontal base excitations have been obtained by Spanos and Koh (1984). The essential results are summarized here for convenient reference and comparison.

Existence

The analytical solution of the rocking response is given by (5a), where a^+, b^+, and ϕ = functions of initial angular displacement and velocity. A sufficient condition for the existence of symmetric periodic response requires: (1) The response period to be equal to the period of excitation or integer multiples of it; and (2) the positive half-cycle of the oscillation to be symmetric to the negative half-cycle. For a given nth subharmonic response, denoted here as a symmetric $(1,n)$ mode, the values of a_s, b_s, and ϕ_s in conjunction with the unknown initial velocity $\dot{\theta}_s$ for the positive half-cycle can be solved

$$\theta^+(\tau) = a_s \sinh \tau + b_s \cosh \tau + 1 + \beta \cos(\Omega\tau + \phi_s) \quad (8)$$

where a_s, b_s, and ϕ_s are functions of e, n, β, and Ω (Spanos and Koh 1984).

The negative half-cycle can be obtained by a time shift and a multiplication factor of minus unity. It is found that either no solution exists, or they always exist in pairs. However, as shown in the following stability analysis, only one solution is stable for some conditions, while the other is always unstable.

Stability

Not all attainable periodic modes presented are stable. The stability of symmetric modes can be determined by taking variations of positive half-cycle response and examining the growth of these variations. Considering the total variations in matrix form, one obtains

$$\begin{bmatrix} \delta\phi_s(\tau_i) \\ \delta\dot{\theta}_s(\tau_i) \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \delta\phi_s \\ \delta\dot{\theta}_s \end{bmatrix} \quad (9)$$

where the coefficients C and A_{jk} are functions of e, n, β, and Ω [see Spanos and Koh (1984) for exact expressions]. For the effect of the initial variations to vanish as the number of impacts increases, the matrix $(\mathbf{A})/C$ must have a spectral radius of less than one. Specifically, the following condition must be satisfied:

$$\left| \frac{(A_{11} + A_{12}) \pm \sqrt{(A_{11} - A_{22})^2 + 4A_{12}A_{21}}}{2C} \right| < 1 \quad (10)$$

This condition can be readily checked upon determining the coefficients C and A_{jk}.

Conditions for Existence of Chaotic Response

In this section, the existence conditions for chaotic response of the rocking system is derived following the methodology outlined by Melnikov (Guckenheimer and Holmes 1985). An integral called the Melnikov function that

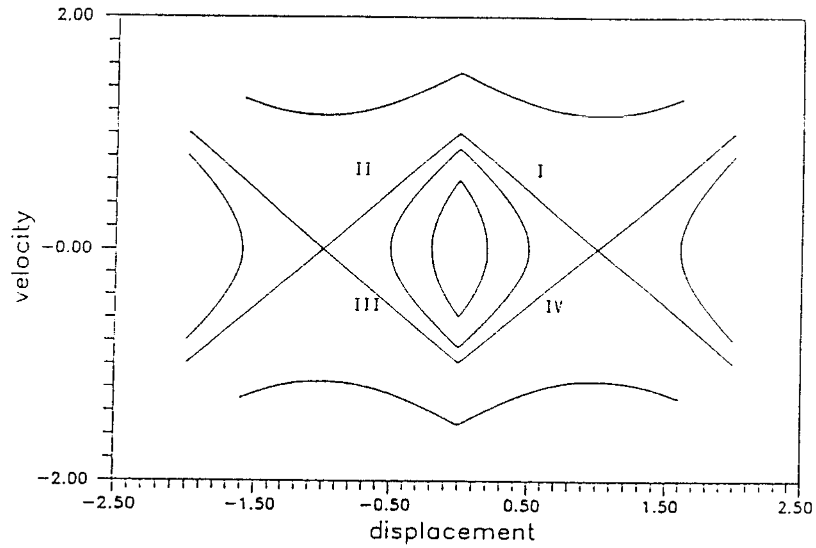

FIG. 9. Hetero-Clinic Orbit for Rocking System

includes the effects of nonlinearity and excitations is constructed. The integration is along a closed curve formed by hetero-clinic orbits that are trajectories connecting two fixed points. The hetero-clinic orbits are identified by examining the phase diagram of the associated (undamped, unforced) Hamiltonian system. The existence of zeros of the Melnikov function indicate the existence of chaotic response. The derivation of the existence conditions is shown as follows.

First, the equation of motion is rewritten in the following vector form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t) \quad (11a)$$

$$\mathbf{x} = (uv)^* \quad (11b)$$

where superscript (*) = the transpose of vector $(\mathbf{u}, \mathbf{v})$. It is assumed that the unperturbed (i.e., undamped and unforced) system has a Hamiltonian H with $f_1 = \partial H/\partial u$ and $f_2 = \partial H/\partial v$, and that the system satisfies the two following conditions:

1. For $\epsilon = 0$, the system possesses a homo-clinic orbit, which is a trajectory connecting a single fixed point to itself, or a pair of hetero-clinic orbits connecting hyperbolic saddle points, forming a closed curve $\mathbf{q}_o(t)$.
2. The interior of $\mathbf{q}_o(t)$ is filled with a continuous family of periodic orbits. When an orbit is close to $\mathbf{q}_o(t)$, its period approaches infinity.

Then there exist two hetero-clinic orbits connecting the two unstable hyperbolic singular points, forming a closed curve. For the rocking system, it can be shown that these conditions are satisfied. The singular points for the rocking object are unstable equilibrium positions, (1,0) and (−1,0), where it stands on one edge (Fig. 9). In the center of the closed curve is a stable singular point (0,0). Phase space trajectories near the stable singular point remain in its neighborhood, while trajectories near the hyperbolic singular points will diverge from them.

The external force and damping are then represented as perturbations to the Hamiltonian system. The damping effect in the rocking system, which causes a sudden reduction in velocity by a factor e at a displacement equal to zero, can be expressed analytically by a function $F(\theta, \dot{\theta})$. The integral of this function along the hetero-clinic orbit is equal to the energy loss caused by impact in each oscillation. For the rocking system considered, the equation of motion for the general case with damping can be written as

$$I_o\ddot{\theta} \pm MgR(\theta_{cr} \mp \theta) = -MRa_{gx}(t) + F(\theta, \dot{\theta}) \quad (12)$$

where $a_{gx} = \epsilon g \cos \omega t$ = the perturbation caused by external force, and the equation with the upper signs represents equation of motion for $\theta > 0$ and the lower signs for $\theta < 0$. The perturbation due to damping is found to be

$$F(\theta, \dot{\theta}) = -\frac{1}{2} I_o\dot{\theta}^2(1 - e^2)\left[\delta\left(\frac{\theta}{\theta_{cr}}\right)\right]\left(\frac{1}{\theta_{cr}}\right) \quad (13)$$

where $\delta(\theta/\theta_{cr})$ = the Dirac delta function.

The equation of motion is then rewritten in the format used in (11) in the form of a first-order system as follows:

$$\dot{\theta} = v \quad (14a)$$

$$\dot{v} = \mp \frac{MgR}{I_o}(\theta_{cr} \mp \theta) - \frac{MgR}{I_o}\epsilon \cos \omega t - \frac{1}{2}v^2(1 - e^2)\delta\left(\frac{\theta}{\theta_{cr}}\right)\left(\frac{1}{\theta_{cr}}\right) \quad (14b)$$

The Hamiltonian function corresponding to the undamped, unforced system is then found by energy considerations

$$H(\theta, v) = \frac{v^2}{2} - \frac{MgR}{2I_o}\theta^2 \pm \frac{MgR}{I_o}\theta\theta_{cr} \quad (15)$$

If the value of the Hamiltonian, which corresponds to the total (kinetic plus potential) energy of the system, exceeds a critical value, $MgR\theta_{cr}^2/2I_o$, the response will become unbounded, leading to overturning. Conversely, if the total energy is less than the critical value, the response will be bounded and stable.

Next, analytical expressions of the hetero-clinic orbits are determined. By equating the Hamiltonian, $H(\theta, v)$, and the critical value, $MgR\theta_{cr}^2/2I_o$, the hetero-clinic orbits can be expressed as functions of time. As mentioned previously (Fig. 9), there are two hetero-clinic orbits. Note that there is a discontinuity within each hetero-clinic orbit, and therefore four analytical expressions are required to describe the closed curve.

Section I (with $\theta > 0$ and $v > 0$) of the hetero-clinic orbit can be expressed as

$$(\theta, v)^1 = (\theta_{cr} - e^{-p}, \alpha e^{-p}) \quad (16a)$$

where $p = \alpha t - \ln \theta_{cr}$. Similarly, section II (with $\theta < 0$ and $v > 0$) can be expressed as

$$(\theta, v)^2 = (e^p - \theta_{cr}, \alpha e^p) \quad (16b)$$

The expressions of the lower orbit (sections III and IV) can be obtained from (16a) and (16b) with opposite signs by symmetry. These expressions are then normalized by setting $\Theta = \theta/\theta_{cr}$, $\alpha t = \tau$, $V = v/\theta_{cr}$, and $\omega = \alpha\Omega$, and rewritten as follows:

$$(\Theta, V)^1 = (1 - e^{-\tau}, e^{-\tau}) \quad (17a)$$

$$(\Theta, V)^2 = (e^{\tau} - 1, e^{\tau}) \quad (17b)$$

$M^+(t_o)$ is used to denote Melnikov's function for the upper orbit, and is defined by

$$M^+(t_o) = \int_{-\infty}^{+\infty} \mathbf{f}[\mathbf{q}_o(t)] \wedge \mathbf{g}[\mathbf{q}_o(t), t + t_o]dt \quad (18a)$$

$$M^+(t_o) = \int_{-\infty}^{+\infty} (f_1 f_2) \wedge (g_1 g_2)^* dt \quad (18b)$$

$$M^+(t_o) = \int_{-\infty}^{+\infty} (f_1 g_2 - f_2 g_1)dt \quad (18c)$$

where the superscript (*) = the transpose of a vector; f_i = the linearized Hamiltonians; and g_i = the perturbations. By symmetry, the expressions for $M^-(t_o)$ and $M^+(t_o)$ are identical. Therefore, the existence of a solution to $M^+(t_o) = 0$ implies the existence of zero solution to $M^-(t_o)$. Thus, the condition for existence of chaotic response can be determined by examining just the upper curve. When no zeros exist, there is no chaotic response for this system.

The analytical methods for the existence and stability of periodic response and the conditions for the existence of chaotic response derived in the preceding are applied to the undamped and damped systems in the following sections.

Response of Undamped Systems

Analytical Predictions

Existence and Stability of Periodic Response

For the undamped system, by setting $e = 1$, the expressions for a_s, b_s, ϕ_s, and $\dot{\Theta}$, can be obtained. However, based on the stability analysis presented in previous sections for any arbitrary combinations of system and excitation parameters with $e = 1.0$, one of the eigenvalues is infinite, while the other is always equal to one. Therefore, (10) is not satisfied, and for all combinations of system and excitation parameters, the periodic responses are unstable, or neutrally stable at best. Thus, the stability analysis method predicts that no stable periodic response exists.

Conditions for Chaotic Response

The Melnikov's function for an undamped slender system can be obtained by letting $F(\theta, v) = 0$, thus, (18) becomes

$$M^+(t_o) = \frac{-2\epsilon\alpha^2 \cos \omega t_o}{\theta_{cr}(\alpha^2 + \omega^2)} \quad (19a)$$

$$M^+(t_o) = -2A_x \frac{\cos \Omega\tau_o}{1 + \Omega^2} \quad (19b)$$

where A_x and Ω = the normalized amplitude and frequency of excitation, respectively, with $A_x = \epsilon/\theta_{cr}$ and $\Omega = \omega/\alpha$. The zero solutions, which yield the condition for the existence of chaotic response, are given by $\cos \Omega\tau_o = 0$. Since this condition can always be satisfied by $\tau_o = 2n\pi$ for all integer n, the condition for the existence of chaotic response is satisfied by any arbitrary combination of system and excitation parameters.

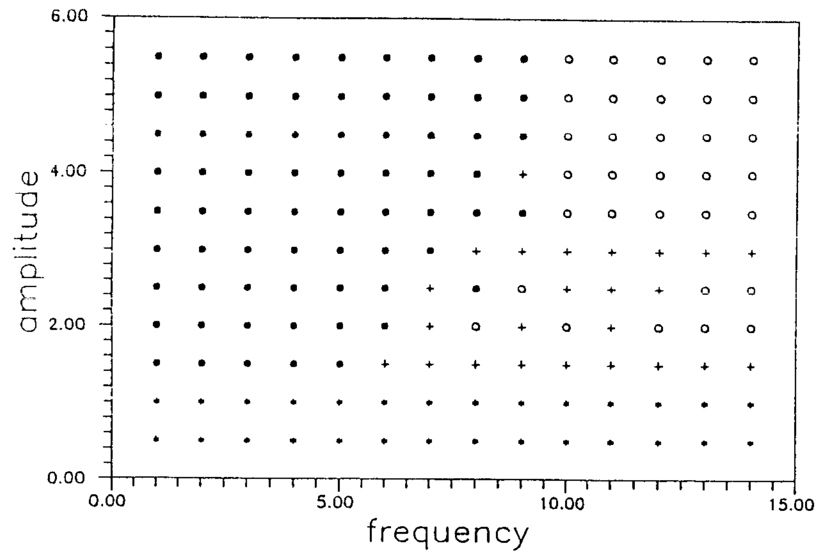

FIG. 10. Parametric Map for Undamped Rocking System (e = 1): Quasi-Periodic Response (○), Chaotic Response (+), No Response (*), and Overturning Response (·)

Numerical Results

Thus, according to stability analysis, there exist no stable periodic responses in the undamped system. Melnikov's function, derived in the preceding section, predicts that for all excitation and system parameters, the conditions for the existence of chaotic response are satisfied, and chaotic and quasi-periodic responses can coexist with the overturning response. These analytical predictions of the behavior of the undamped systems are confirmed by numerical results. As shown in the parametric map in Fig. 10, there are no stable periodic rocking responses, only chaotic and quasi-periodic responses coexisting with the overturning response.

The time history, phase diagram, Poincaré map, and spectral density of a typical chaotic response of an undamped system have been shown in Figs. 6, 7, and 8, respectively. The system and excitation parameters are $e = 1.0$, $A_x = 2.0$, and $T = 0.4$. The Lyapunov exponent and fractal dimension for this particular response are 0.14 and 2.3, respectively, thus confirming the chaotic nature of the response.

Quasi-periodic responses may be considered a transition between periodic and chaotic response. They usually appear in conjunction with chaotic responses; thus, their appearance seems to be indicative of imminent chaotic response. However, no clear pattern shows the conditions of existence for quasi-periodic response, although it exists in the undamped system. The time history, phase diagram, Poincaré map, and spectral density of a typical quasi-periodic response of an undamped system have been shown in Figs. 5, 7, and 8, respectively. The system and excitation parameters are $e = 1.0$, $A_x = 4.0$, and $T = 0.4$. As expected, the Lyapunov exponent and fractal dimension are 0.0 and 2.0, respectively, confirming the quasi-periodic nature of the response.

Thus, based on numerical results, it is observed that instability of periodic response does not necessarily imply overturning. For undamped slender systems, no periodic response exists; instead, there exist two other bounded responses—quasi-periodic and chaotic. Stability analysis and Melnikov analysis predictions are consistent with numerical results.

The individual rocking motion about an edge, although linear, contains exponentially divergent terms and by itself is unbounded [(5*a*) and (5*b*)]. Thus, small perturbations in the initial conditions of rocking motions about an edge will diverge as time increases. However, the transition nonlinearity at impact of the undamped rocking system provides the "folding" needed for the nonlinear (combined) rocking response to become bounded. These two elements together provide the ingredients for bounded chaotic responses. Since chaotic responses are characterized by extreme sensitivity to system and excitation parameters, it can be concluded that the exponential divergent nature and the transition nonlinearity together form a major cause of extreme sensitivity of the rocking response.

Response of Damped Slender Systems

Analytical Predictions

Existence and Stability of Periodic Response

For damped slender systems, the solution for periodic response is given by (8). The condition for stability of the periodic responses with various system and excitation parameters is given by (10). The stability curves for (1,1) and (1,3) modes for a typical damped system with $e = 0.925$ are shown in Fig. 11. As discussed by Spanos and Koh (1984), the stability of rocking modes decreases with increasing mode number.

Conditions for Chaotic Response

For damped systems, the Melnikov's function derived in previous sections [(18)] can be written as follows:

$$M^+(t_o) = \int_{-\infty}^{+\infty} v^{(2)}(t)[\epsilon\alpha^2 \cos \omega(t + t_o) - F(\theta, \dot{\theta})]dt$$

$$+ \int_{-\infty}^{+\infty} v^{(1)}(t)[\epsilon\alpha^2 \cos \omega(t + t_o) - F(\theta, \dot{\theta})]dt \quad \text{(20}a\text{)}$$

$$M^+(t_o) = \frac{-2\epsilon\alpha^2 \cos \omega t_o}{\alpha^2 + \omega^2} - \frac{\theta_{cr}(1 - e^2)}{2} \quad \text{(20}b\text{)}$$

$M^+(t_o)$ equal to zero is equivalent to

$$\cos \Omega\tau_o = -\frac{(1 - e^2)(1 + \Omega^2)}{4A_x} \quad (21)$$

where A_x and Ω = the normalized amplitude and frequency of excitation, respectively, with $A_x = \epsilon/\theta_{cr}$ and $\Omega = \omega/\alpha$.

Eq. (21) can always be satisfied by some τ_o only if the magnitude of the right-hand side is less than or equal to unity. Thus, the existence of chaotic response is given by the combination of system parameters, e, Ω, and A_x that satisfy the following inequality:

$$\frac{(1 - e^2)(1 + \Omega^2)}{4A_x} \leq 1 \quad (22)$$

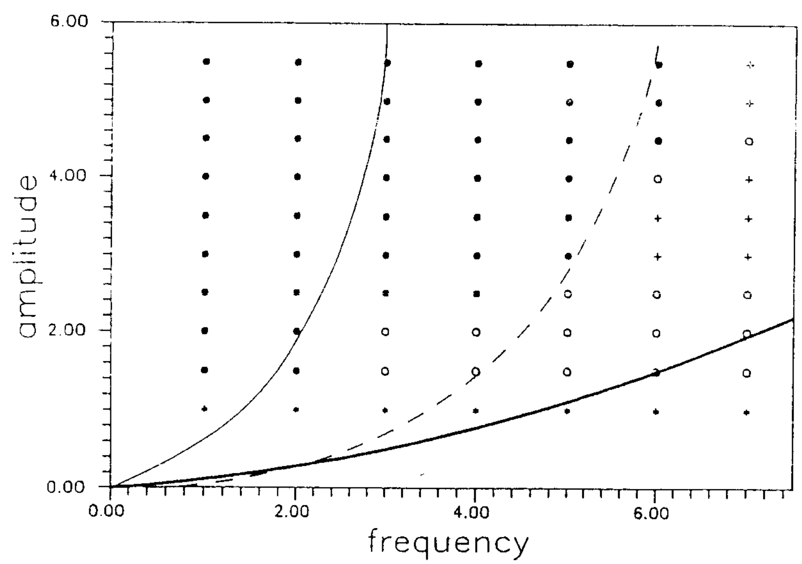

FIG. 11. Theoretical Prediction and Parametric Map for Damped Rocking System (e = 0.925): Upper Bound for Harmonic Response (—); Upper Bound for One-Third Subharmonic Response (- -); Lower Bound for Chaotic Response (—); Harmonic Response (○); One-Third Subharmonic Response (+); No Response (*); and Overturning Response (·)

Numerical Results

To determine the accuracy of the analytical predictions given by stability and Melnikov analyses, a numerical parametric study has been conducted. Results show that the most predominantly stable mode observed, starting from a quiescent position, is the symmetric (1,1) mode, a typical example of which has been shown in Fig. 2(*a*). The second most common mode is the subharmonic (1,3) mode [Fig. 2(*b*)]. Several unsymmetric, even-order modes have been found; a sample is shown in Fig. 2(*c*). However, the existence of unsymmetric modes cannot be predicted by stability analysis, but can only be determined using numerical simulation. No particular pattern of occurrence in these unsymmetric modes was observed.

The parametric map shown in Fig. 11 indicates that the analytical stability boundary given by (10) is reasonably accurate. Although these analytical predictions are restricted to symmetric responses, the symmetric odd-order modes are the dominant modes. Unstable modes predicted by analysis do not necessarily imply overturning of the object. The actual responses may converge to other stable periodic modes.

As shown in Fig. 11, in the region where stability analysis predicts periodic response and Melnikov analysis predicts possible chaotic response, with quiescent initial conditions, only periodic and overturning responses were found. This result is consistent with those obtained by Spanos and Koh (1984). Under an extensive numerical search (in a multidimensional space), by varying the initial values of the angular displacement and velocity as well as the system parameters, chaotic and quasi-periodic responses are found to exist. However, the existence of these bounded responses are rare and dif-

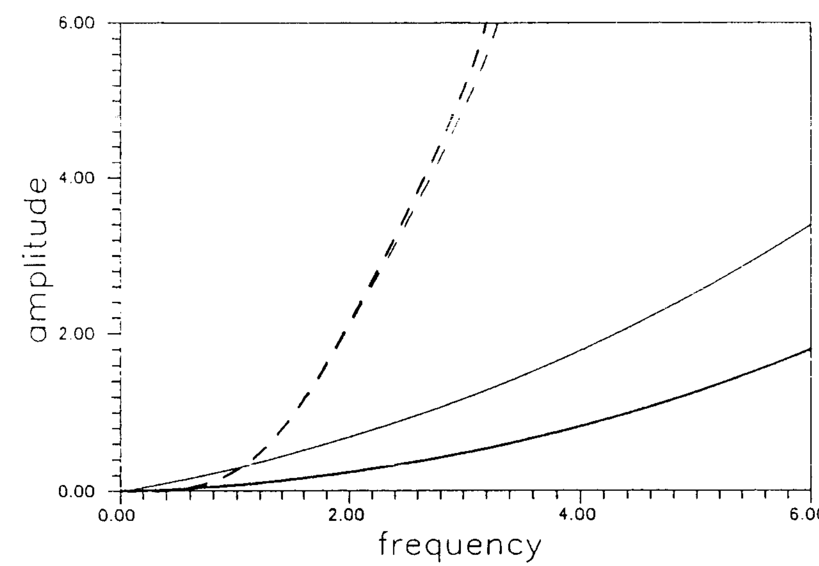

FIG. 12. Theoretical Predictions for Damped Rocking System: Upper Bound for Harmonic Response with e = 0.8 (--); Upper Bound for Harmonic Response with e = 0.9 (--); Lower Bound for Chaotic Response with e = 0.8 (—); and Lower Bound for Chaotic Response with e = 0.9 (—)

ficult to detect, with no apparent identifiable pattern. Thus, it appears that, for damped rocking systems, the Melnikov analysis method yields a necessary condition (a lower bound) for the existence of a chaotic response, but the condition may be insufficient to guarantee the existence of these responses. Consistent with the studies of other nonlinear systems, the Melnikov boundary provides only guidance for locating chaotic responses.

As shown in Fig. 12, the region bounded by the stable (1,1) mode and the Melnikov chaotic boundary expands as the coefficient of restitution e increases toward unity, suggesting that chaotic response is more likely to occur for systems with small energy dissipation (lightly damped). Indeed, based on an extensive parametric search, the chaotic responses detected are from systems with e greater than 0.95.

Comparing the parametric maps for the undamped and the damped cases (Figs. 10 and 11, respectively), it is observed that energy dissipation induces a large region of stable periodic modes and reduces the region of chaotic responses from the entire parametric space to merely scattered spots in the space. Since periodic responses are relatively insensitive to perturbations of system and excitation parameters, it may be concluded that the energy dissipation nonlinearity reduces the sensitivity of the rocking response.

Braced System

To further clarify the influence of impact transition, the response behavior of a system with rigid braces at the critical angle subjected to horizontal excitation is examined. For the given brace angle, responses with normalized displacement amplitude less than unity are not affected by the presence of the braces. When the rotation angle reaches the critical angle, perfect elastic impact occurs. Results show that for those responses with frequent side-

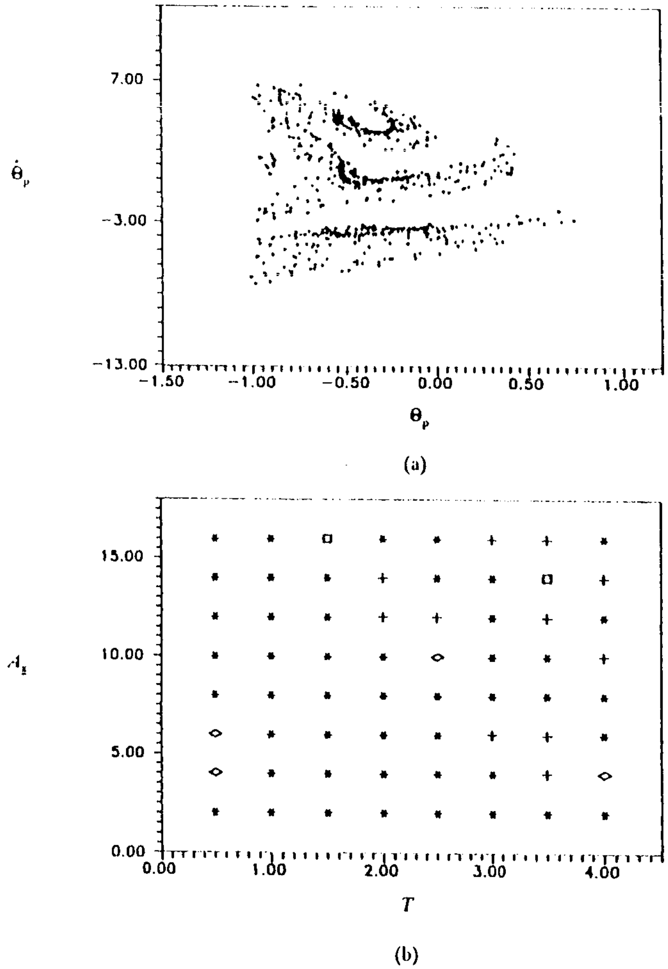

FIG. 13. Side-Braced System: (*a*) Poincaré Map, A_x = 15, T = 2; (*b*) Parametric Map, e = 0.925, Chaotic (+), Harmonic (*), One-Third Subharmonic (◇), and Even-Order Subharmonic (□)

brace impacts, chaotic response dominates. The chaotic attractor of a typical chaotic response is shown in Fig. 13(*a*). Its characteristics are quite different from those with no side braces. The parametric map, shown in Fig. 13(*b*), shows that chaotic responses appear frequently, but their occurrence is scattered. Unsymmetric, even modes also occur with the dominant symmetric (1, 1) mode. Thus transition nonlinearity induces complex and chaotic responses and is an important source of sensitivity of the rocking response.

Summary and Conclusions

This investigation deals with the chaotic behavior of the rocking response of slender rigid objects to harmonic base excitations. The existence and stability of periodic, chaotic, and quasi-periodic responses are examined.

An analytical technique based on the Melnikov function is derived for the prediction of the existence of chaotic response. The accuracy of these predictions has been assessed by numerical simulation results. The effects of two causes of nonlinearity, the transition of governing equations and the abrupt reduction in angular velocity due to energy dissipation at impact, were examined in detail.

A study of the analytical predictions and numerical results leads to the following conclusions:

1. Rocking behavior is much more complicated than the periodic and overturning responses studied thus far in the literature. The transition nonlinearity at impact induces responses that cannot be predicted by classical stability analysis alone. With modern analytical and numerical methods, two additional types of steady-state responses were discovered in this study. It was shown that for some combinations of system and excitation parameters, quasi-periodic and chaotic responses may coexist with periodic and overturning responses.
2. Contrary to previous belief, the lack of stable periodic responses does not necessary imply overturning; quasi-periodic and chaotic responses may result. The analysis technique derived in this study can provide a boundary for the existence of chaotic response.
3. For an undamped system, the only nonlinearity is caused by the transition of the governing equation of motion at impact. Quasi-periodic, chaotic, and overturning responses, which are characterized by extreme sensitivity to initial conditions, are the only stable responses; periodic response does not exist. Thus, transition at impact, together with the exponentially divergent nature of the individual rocking motions, constitutes a major cause of extreme sensitivity of the rocking response. Furthermore, when side bracing is introduced, the responses become more sensitive, and the region of chaotic response is enlarged, confirming transition nonlinearity as a major source of sensitivity.
4. Nonlinearity due to energy dissipation decreases the sensitivity of the system. When energy dissipation at impact in the form of an abrupt reduction in angular velocity is introduced, periodic response becomes dominant. Numerical results indicate that chaotic and quasi-periodic responses occur only rarely.

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Appendix II. Notation

The following symbols are used in this paper:

- A_x = normalized amplitude of horizontal excitation;
- a_{gx} = horizontal ground acceleration;
- a_x = coefficient of periodic steady-state response;
- a_x = amplitude of horizontal excitation;
- b_x = coefficient of periodic steady-state response;
- e = coefficient of restitution;
- g = acceleration of gravity;
- I_o = mass moment of inertia;
- M = mass of object;
- R = radius of rotation;
- T = period of excitation;
- α = $(MgR/I_o)^{1/2}$;
- β = $A_x/(1 + \Omega^2)$;
- Θ = normalized rotation angle; $\Theta = \theta/\theta_{cr}$;
- $\dot{\Theta}$ = normalized angular velocity;
- θ = rotation angle of rocking object;
- θ_{cr} = critical angle;
- $\dot{\theta}$ = angular velocity;
- τ = initial time, $\tau = \alpha t$;
- τ_i = instant of occurrence of impact;
- ϕ = phase shift of horizontal excitation;
- ϕ_x = shift between excitation and steady-state response;
- Ω = normalized frequency of horizontal excitation; and
- ω = frequency of horizontal excitation.