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Analysis of Nonlinear Response of an Articulated Tower

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ABSTRACT

A semi-analytical method is employed to investigate stability of the nonlinear response of an articulated tower. Local and global bifurcations determine the possible existence of complex nonlinear and chaotic motions which cannot be obtained through evaluation of an equivalent linearized system.

INTRODUCTION

Complex nonlinear and chaotic responses have been recently observed in various models of articulated towers and other compliant ocean systems (e.g., Thompson et al., 1984; Liaw, 1988). Similar behavior has been found in roll response of ships and semisubmersibles where the restoring moment was modeled by a quintic polynomial (Nayfeh and Khdeir, 1986; Witz et al., 1989). Articulated towers are surface piercing columns pinned to the sea floor which serve as mooring loading terminals for oil tankers. They are characterized by a nonlinear restoring moment and a nonlinear coupled hydrodynamic exciting moment. The restoring moment of the articulated tower is that of a forced plane pendulum and is generated by an internal excess buoyancy mechanism. The exciting moment includes a coupled wave-structure viscous drag component and a wave induced inertial moment. The drag component consists of parametric and quadratic damping, a bias and harmonic forcing.

The forced pendulum has been extensively investigated and complex nonlinear behavior, such as coexistence of attractors, symmetry breaking, period doubling and intermittency has been found experimentally, numerically and analytically (D'Humiers et al., 1982; Miles, 1988). Furthermore, global asymptotic criteria for the existence of chaotic response have been derived for the pendulum and for a Josephson junction circuit (Salam and Sastry, 1985) modeled as a forced and biased pendulum. However, unlike the unperturbed pendulum which has a pair of homoclinic orbits separating the domain of response into two disjoint parts of bounded and unbounded solutions, the articulated tower belongs to a family of oscillators which have a unique equilibrium position.

While weakly nonlinear systems have been studied extensively from both classical (Nayfeh and Mook, 1979) and modern approaches (Guckenheimer and Holmes, 1983), complex single equilibrium point systems are limited in their scope of analysis. Examples of these systems are the hardening Duffing equation analyzed by modified multiple scales and by the method of harmonic balance (Rahman and Burton, 1986; Szemplinska-Stupnicka, 1987) and the subharmonic motions of a wind loaded structure analyzed by the general method of averaging (Holmes, 1980). Stability analysis of system behavior results in local bifurcation maps defining regions of existence of the various nonlinear phenomena in parameter space. This analysis consists of perturbing the approximate solution and analyzing the resulting variational equation numerically by Floquet analysis or by analytically solving the equivalent Hill's variational equation. Both methods have been successfully employed on the hardening and softening Duffing equation (Szemplinska-Stupnicka, 1988; Nayfeh and Sanchez, 1989).

Extensive investigation of the articulated tower model has been performed for various configurations of slender and non-slender towers (e.g., Patel, 1989; Chakrabarti, 1987) in which the response was assumed small and the nonlinear moments were equivalently linearized by various methods. However, equivalent linearization eliminates the possibility of obtaining coexisting solutions and other nonlinear phenomena. Various configurations of articulated towers moored to floating structures have revealed the existence of subharmonic and chaotic response. Three models describing these motions are a bilinear oscillator identifying a stiffness discontinuity due to slackening mooring lines (Thompson et al., 1984), and two nonlinear oscillators where the restoring moment is characterized by cubic (Choi & Lou, 1991) and quartic (Fujino & Sagara, 1990) polynomials respectively. These models assumed small amplitude response and the restoring moment consisted of a linearized buoyancy component and the complementary nonlinear mooring stiffness function. The hydrodynamic drag moment was simplified to a quadratic damping function in the latter model and was linearized in the former models.

This paper describes a semi-analytic stability analysis performed on the nonlinear response of an unconstrained slender articulated tower. Consequently, the restoring moment consists of only a nonlinear buoyancy component. In order to model the nonlinear wave-structure coupling effect, the exact relative motion quadratic drag component is retained. Thus, the predicted local and global bifurcations determine the complex nonlinear behavior found numerically and can serve as a reference for identification of the generating mechanisms of instabilities in models with nonlinear mooring functions.

MODEL FORMULATION

The articulated tower considered (Fig. 1) is modeled as a single degree of freedom (θ: pitch), hydrodynamically damped and excited nonlinear oscillator. The equation of motion is modeled by a relative motion Morison equation with frequency independent coefficients (Sarpkaya and Issacson, 1983). This

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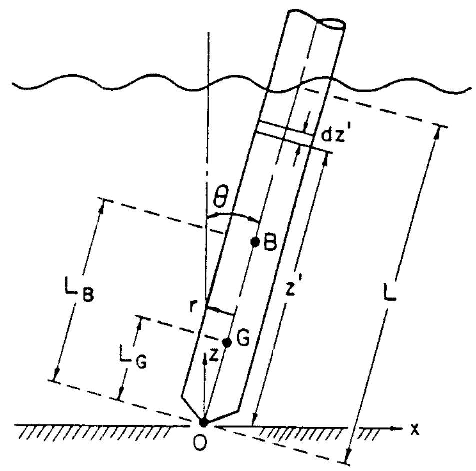

Fig. 1 Definition sketch

approach, which assumes that the presence of the structure does not affect the wave field, hence waves propagating past the structure remain unmodified, can be justified for slender body motion in the vertical plane (surge, heave, pitch) where the wavelength is large compared to the beam of the structure (Newman, 1977).

The restoring moment is calculated about the base of the tower [$R(\theta)$] and friction at the joint is included with small structural damping (C). The exciting moment [$M(\theta;t)$] is calculated along a stationary upright tower assuming linear wave kinematics.

$$I\ddot{\theta}+C\dot{\theta}+R(\theta) = M(\ddot{\theta},\dot{\theta};t) \tag{1a}$$

where

$$R(\theta) = (F_B L_B - F_W L_G)\sin\theta \tag{1b}$$

$$M(\ddot{\theta},\dot{\theta},t) = \int_0^h dM_I(z')+dM_D(z')+dM_{FK}(z') \tag{1c}$$

$dM_{I,D,FK}$ = differential moment components (inertia, drag, Froude-Krilov):

$$\begin{aligned} dM_I &= \rho C_a \pi r^2 z'[\dot{U}(z')-z'\ddot{\theta}]dz' \\ dM_D &= \rho C_d r\, z'[U(z')-z'\dot{\theta}]\left|U(z')-z'\dot{\theta}\right|dz' \\ dM_{FK} &= \rho\, \pi r^2 z'[\dot{U}(z')]\, dz' \end{aligned} \tag{1d}$$

$$U(z') = \omega a\frac{\cosh kz'}{\sinh kh}\cos\omega t \tag{1e}$$

and

a, ω, k = wave amplitude, frequency and number ($\omega^2 = gk \tanh kh$)
C_d, C_a = drag and inertia coefficients
F_B, F_W = buoyancy force ($\pi\rho g r^2 h$), tower weight
I, C = moment of inertia, structural damping
r, h = radius of structure, submerged length
$[\,\dot{}\,]$ = differentiation with respect to time

Rearranging and normalizing ($s'=kz'$) the equation of motion (Eq. 1) yields the following nonlinear oscillator:

$$\ddot{\theta}+\gamma\dot{\theta}+R(\theta)=M_D(\dot{\theta};t)-F(t) \tag{2a}$$

where

$$R(\theta)=\alpha\sin\theta \tag{2b}$$

$$F(t)=\mu k_\mu f\sin\omega t \tag{2c}$$

$$M_D(\dot{\theta},t)=\delta k_\delta\int_0^{kh} s'[u-s'\frac{\dot{\theta}}{\omega}]\left|u-s'\frac{\dot{\theta}}{\omega}\right|ds' \tag{2d}$$

$$U(s')=f\frac{\cosh s'}{\sinh kh}\cos\omega t \tag{2e}$$

and

$$\alpha=\frac{3}{C_a}\frac{L_B}{h}\frac{1-F_W L_G/F_B L_B}{1+I/I_a}\frac{g}{h}\ ,\quad \gamma=\frac{C}{I+I_a}$$

$$\delta=\frac{3}{\pi}\frac{C_d/C_a}{1+I/I_a}\frac{g}{h}\ ,\quad \mu=3\frac{1+1/C_a}{1+I/I_a}\frac{g}{h}$$

$$\kappa_\delta=\frac{\tanh kh}{kr(kh)^2}\ ,\quad \kappa_\mu=\frac{(kh)\sinh kh-\cosh kh+1}{(kh)^2\tanh kh}$$

$$f=ka\ ,\quad I_a=\frac{\pi}{3}C_a\rho r^2 h^3$$

Note that $kr < \pi/5$ (diffraction parameter), $ka < \pi/7$ (wave steepness parameter).

The potential of the undamped unforced (Hamiltonian) system ($M_D, F, \gamma=0$) can be obtained by integrating the restoring force resulting with an invariant quantity for the Hamiltonian energy. However, unlike the unperturbed pendulum, the potential well has no maxima and the Hamiltonian energy is bounded [$H(\theta, d\theta/dt)<2$: oscillations].

$$H(\theta,\dot{\theta})=\frac{\dot{\theta}^2}{2}+\alpha(1-\cos\theta) \tag{3}$$

HARMONIC AND SUBHARMONIC SOLUTIONS

Investigation of system response is similar to analysis of a system with an unsymmetric elastic stiffness function. Loss of symmetry is due to a bias caused by the wave-induced drag moment. In both harmonic and subharmonic response the nT periodic solution can be approximated by a finite Fourier series expansion.

$$\theta_n(t)\cong A_0+\sum_{m=1}^{M}A_{m/n}\cos(m\frac{\omega t}{n}+\Phi_{m/n}) \tag{4}$$

where $A_0, A_{m/n}, \Phi_{m/n}$ - solution amplitudes and phases.

and $m=1,2,3,...,M$; M is the order of approximation.
$n=1,2,3,...,N$; N is the order of subharmonic.

An unsymmetric solution [$\theta_0(t) \neq -\theta_0(t+nT/2)$] includes both even and odd harmonics whereas a symmetric solution would consist of only odd harmonics.

Due to the complex form of the drag nonlinearity, the method of harmonic balance is employed (Hayashi, 1964). The approximate solution (Eq. 4) is substituted into the system (Eq. 2). After rearranging, squaring and calculating the differential drag moment along the tower, the harmonic and constant coefficients are equated separately to zero. Thus, the original nonlinear ordinary differential equation is replaced by a finite set of algebraic equations.

$$G_i(A_o, A_m, \Phi_m)=0 \quad (5)$$

where $i = 1,2,...,I_T$; $I_T = 2M+1$ is the size of the set.

See Appendix (A.1) for detail of a low-order set (I_T=3) corresponding to $\theta = A_0 + A_1\cos(\omega t+\Phi_1)$.

Solutions of this set are obtained using an iterative Newton-Raphson procedure from which frequency response curves are generated numerically by solving for the unknown amplitudes (A_0,A_m) and phases (Φ_m). Note that for very small pitch angles [$R(\theta)\rightarrow\theta$] the anticipated first order solution is that of a biased linear oscillator:

$$A_o^2=\frac{3}{8}\left(\frac{\delta}{\alpha}\right)^2(2S_0^2+S_{2S}^2+S_{2C}^2) \quad (6a)$$

$$A_1^2=\frac{(\mu\kappa_u f)^2}{(\alpha-\omega^2)^2+(\gamma\omega)^2} \quad (6b)$$

$$\Phi_1=\tan^{-1}\left(-\frac{\alpha-\omega^2}{\gamma\omega}\right) \quad (6c)$$

STABILITY ANALYSIS

Global stability of the system is performed by a Lyapunov function [$V(\theta, d\theta/dt)$] approach (Hagedorn, 1978). The Hamiltonian energy (Eq.3) is a weak Lyapunov function [$V(0,0)=0$ and $dV/dt \equiv 0$] resulting in neutral stable solutions (centers):

$$\dot\theta=\sqrt{2[h-\alpha(1-\cos\theta)]} \quad (7)$$

where $h=H(\theta,d\theta/dt)$ is calculated from initial conditions.

Modification of $V(\theta,d\theta/dt)$ to account for structural damping results in a strong Lyapunov function:

$$V(\theta,\dot\theta)=\frac{\dot\theta^2}{2}+\alpha(1-\cos\theta)+\upsilon(\theta\dot\theta+\gamma\frac{\theta^2}{2}) \quad (8a)$$

and

$$\dot V=-\upsilon[\theta R(\theta)]-(\gamma-\upsilon)\dot\theta^2 \quad (8b)$$

Choosing ν sufficiently small ($0<\nu<\gamma$) results in a globally stable unforced system [i.e., $V(\theta,d\theta/dt)$ positive definite and $dV/dt\le0$) describing in phase plane (θ,dθ/dt) an asymptotically stable hyperbolic fixed point (sink) at the origin. However, with the addition of wave excitation the sink becomes a hyperbolic closed orbit (limit cycle). The limit cycle loses the circularity of the sink but is anticipated by the invariant manifold theorem to retain its stable characteristics (Guckenheimer and Holmes, 1986). Although this result ensures that solutions remain bounded for small excitation (M_D,F<<1), investigation of the influence of larger excitation can only be performed by local analysis (Nayfeh and Mook, 1979).

Local stability is determined by considering a perturbed solution $\theta(t)=\theta_0(t)+\varepsilon(t)$, where $\theta_0(t)$ is the approximate solution and $\varepsilon(t)$ is a small perturbation. Substituting $\theta(t)$ in Eq.2 and simplifying the resulting equation lead to a nonlinear variational equation.

$$\ddot\varepsilon+D[\dot\varepsilon;\dot\theta_o(t)]+G[\varepsilon;\theta_o(t)]=0 \quad (9a)$$

where

$$G(\varepsilon)=\alpha[\sin(\theta_o+\varepsilon)-\sin\theta_o] \quad (9b)$$

$$D(\dot\varepsilon)=\gamma\dot\varepsilon+M_D(\theta_o+\dot\varepsilon;t)-M_D(\theta_o;t) \quad (9c)$$

Linearizing the variational (Eqs. 9b, c) yields a linear ordinary differential equation with a periodic coefficient functions $H_{R,D}[\theta_0(t)]=H_{R,D}[\theta_0(t+T)]$.

$$\ddot\varepsilon+H_D[\dot\theta_o(t)]\dot\varepsilon+H_R[\theta_o(t)]\varepsilon=0 \quad (10a)$$

where

$$H_R(\theta_o)=\alpha\cos\theta_o \quad (10b)$$

$$H_D(\dot\theta_o)=\gamma+2\frac{\delta\kappa_\delta}{\omega}\int_o^{kh}(s')^2\left|u(s')-s'\dot\theta_o\right|ds' \quad (10c)$$

Substitution of the solution Eq. 4 in Eq. 10 and expanding $H_{D,R}(\theta_0)$ in Fourier series $H_{D,R}(\psi)$; ($\psi=\omega t+\Phi_1$) lead to a general Hill's variational equation.

$$\ddot\varepsilon+H_D[\psi(t)]\dot\varepsilon+H_R[\psi(t)]\varepsilon=0 \quad (11a)$$

where

$$H_R(\psi)=\alpha[\lambda_o+\sum_{j=1}^{\infty}\lambda_j\cos j\psi] \quad (11b)$$

$$H_D(\psi)=\gamma+2\frac{\delta}{\omega}\left|\mu_{1C}\cos\psi+\mu_{1S}\sin\psi\right| \quad (11c)$$

See Appendix (A.2) for λ, μ.

The particular solution to Eq. 11 is $\varepsilon(t)=\exp(\zeta t)Z(t)$. Application of Floquet theory (Ioos and Joseph, 1981) yields two forms of the particular solution: $Z(t)=Z(t+T)$ and $Z(t)=Z(t+2T)$ which are due to the odd and even terms in Eq. 4, respectively. Thus, even a low order two-term solution [$\theta_0(t)=A_0+A_1\cos(\psi)$] defines two unstable regions. The first unstable region [$Z(t+T)$] is identified by the cos(2ψ) term in Eq. 11b and coincides with the vertical tangent points of the primary resonance on the frequency response curve. However, the constant amplitude (A_0) generates a cos(ψ) in Eq. 11b and 11c and defines the lowest order unstable region [$Z(t)=Z(t+2T)$] or secondary resonance.

The boundaries of the unstable regions are obtained by inserting the following solution forms into Eq. 11 and applying the method of harmonic balance at the stability limit (ζ=0).

$$\varepsilon(t)=b_o+b_1\cos(\omega t+\Phi_1)\,;\,Z(t+T)$$

$$\varepsilon(t)=b_{1/2}\cos(\frac{\omega t}{2}+\Phi_{1/2})\,;\,Z(t+2T) \quad (12)$$

The condition for a non-zero solution results in a determinant [$\Delta(\omega)=0$] from which the boundaries of the secondary resonance can be obtained [$\Delta(\omega)<0$ for $\zeta>0$: unstable]. The unstable region is defined by the intersection of the stability curves and the frequency response curve (Fig. 2 : nonlinear restoring moment).

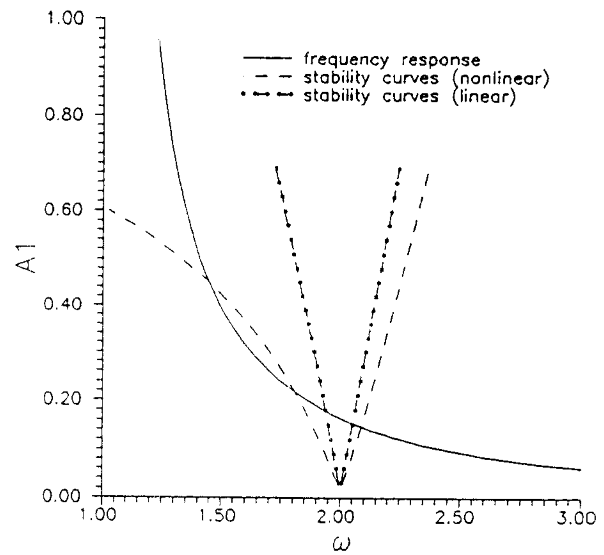

Fig. 2 Stability loss of the T periodic response [A1 (radian) vs. ω (radian/s)]
[α=0.5, γ=0.01, δ=0.1, f=ka=0.1, μ=2, kh=π, kr=0.5]

$$\Delta(\omega)=\left(\alpha\lambda_o-\frac{\omega^2}{4}\right)^2+\left(\frac{\gamma\omega}{2}\right)^2-\left(\frac{\alpha\lambda_1}{2}\right)^2 +sgn(\chi)\left(\frac{\alpha\delta}{2}\right)\lambda_1\mu_{1S}-\left(\frac{\delta}{2}\right)^2(\mu_{1C}^2+\mu_{1S}^2) \quad (13)$$

where $\chi=(\mu_{1C}\cos\Phi_1 + \mu_{1S}\sin\Phi_1)$ and sgn(χ) denotes the sign of χ.

For small pitch angles [$R(\theta)\rightarrow\theta$] Eq. 13 can be further simplified (λ_0=1, λ_1=0) resulting in a narrower unstable region (Fig. 2 : linear restoring moment).

$$\Delta(\omega)=\left(\alpha-\frac{\omega^2}{4}\right)^2+\left(\frac{\gamma\omega}{2}\right)^2-\left(\frac{\delta}{2}\right)^2(\mu_{1C}^2+\mu_{1S}^2) \quad (14)$$

Thus, the unstable region defining the secondary resonance can be shown to be confined between two hyperbolic functions [$\Delta(\omega)=0$, $\gamma<<\delta$] and is shown to be sensitive to the magnitude of the response as indicated by:

$$A_1\propto\frac{1}{2\delta}\left|1-\frac{4\alpha}{\omega^2}\right| \quad (15)$$

Note that the undamped system (γ,δ=0) Eq. 13 simplifies to:

$$\omega^2 = 4\alpha\,(\lambda_0 - \lambda_1/2).$$

LOCAL AND GLOBAL BIFURCATIONS

The variational equation (Eq. 11) reveals two regions where the T periodic solution loses its stability. The primary and secondary resonances are defined by saddle-node (tangent) and period doubling (flip) bifurcations respectively. These bifurcations are local but they reveal coexistence of solutions. The tangent bifurcations form the jump phenomena in which two coexisting stable solutions are separated by an unstable solution. Unsymmetric subharmonics of period nT will occur in multiple "partner" orbits (Holmes, 1979). Note that coexistence of solutions is defined by different initial conditions in the same parameter space.

In order to investigate further the stability of the 1/2 subharmonic (Eq. 4: n=2), the solution is perturbed [$\theta'(t)=\theta'_0(t)+\varepsilon'(t)$; where ' denotes the 2T solution] to obtain the subharmonic variational equation [$\psi'=\psi(\omega t/2+\Phi 1/2\ ;\ \omega t+\Phi_1)$].

$$\ddot{\varepsilon}+H_D[\psi'(\frac{t}{2})]\dot{\varepsilon}+H_R[\psi'(\frac{t}{2})]\varepsilon=0 \quad (16a)$$

where

$$H_R(\psi')=\alpha[\lambda_o+\sum_{j=1}^{\infty}\lambda_{jC/2}\cos j\frac{\omega t}{2}+\lambda_{jS/2}\sin j\frac{\omega t}{2}] \quad (16b)$$

$$H_D(\psi')=\gamma+2\frac{\delta}{\omega}\left|\sum_{j=1}^{2}\mu_{jC/2}\cos j\frac{\omega t}{2}+\mu_{jS/2}\sin j\frac{\omega t}{2}\right| \quad (16c)$$

See Appendix (A.3) for λ', μ'.

Similarly to the stability analysis in the previous section, a low-order three-term solution (Eq. 4: N=2, M=2):

$$\theta_0(t) = A_0 + A_{1/2}\cos(\omega t/2+\Phi_{1/2}) + A_1\cos(\omega t+\Phi_1)$$

generates two unstable regions. The first order unstable region [$Z(t+2T)$] is identified by the cos(ωt) term in Eq. 16 whereas the lowest order unstable region [$Z(t+4T)$] is identified by cos(ωt/2). These instabilities are associated with the tangent and the flip bifurcations which cause them.

Thus, the general Hill's equation suggests the possible cascade of period doubling bifurcations. If the period doubling sequence is infinite, the resulting motion is chaotic (Thompson and Stewart, 1986).

System responses obtained by numerically integrating the equation of motion Eq. 2 show the evolution of a period doubling cascade (Figs. 3 and 4). The results are portrayed in phase planes ($\theta,d\theta/dt$) and Poincaré maps ($\theta_p,d\theta_p/dt$) which are sampled at the forcing period ($T=2\pi/\omega$). An nT subharmonic will repeat after n intervals (Fig. 3a: $2T$) and the chaotic attractor will generate a fractal map (Fig. 3b: 2^nT). The power spectra of the response [$S_\theta(\omega)$] also depicts the period doubling (Fig. 4a) and is continuous, showing "random like" behavior when the response becomes chaotic (Fig. 4b). Note that the chaotic attractor is sensitive to initial conditions and can coexist with other steady state solutions.

SUMMARY AND CONCLUSIONS

Stability of approximate low-order periodic solutions enables the analysis of the nonlinearities governing the complex response of the articulated tower ocean system.

The system is modeled by the exact forms of the restoring and hydrodynamic exciting moment. A semi-analytic method is applied to the system resulting in local and global bifurcations. The method consists of intersecting approximated system response with stability curves derived from a generalized Hill's variational equation. Consequently, local bifurcations determine

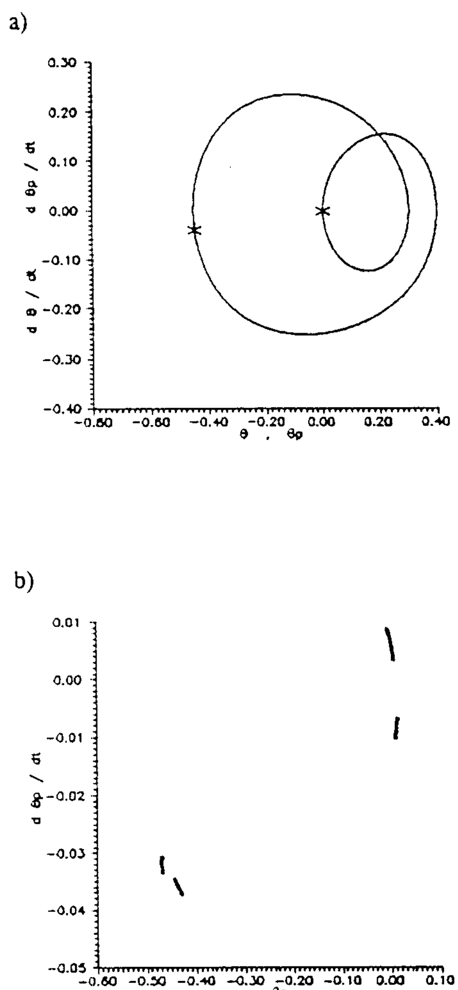

Fig. 3 Period doubling cascade: phase plane & Poincaré map [$d\theta/dt$ & $d\theta_p/dt$ (radian/sec) vs. θ & θ_p (radian)] a) $2T$ (ω=1.5), b) 2^nT (ω=1.54) [α=0.5, γ=0.01, δ=0.1, μ=2, f=ka=0.1, kh=π, kr=0.5]

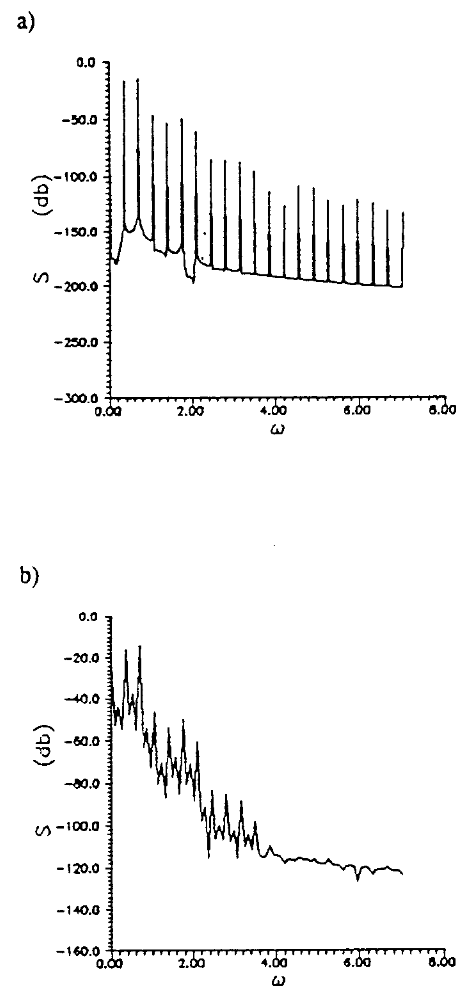

Fig. 4 Period doubling cascade: power spectra [$S_\theta(\omega)$ (db) vs. ω (radian/sec)] a) $2T$ (ω=1.5), b) 2^nT (ω=1.54) [α=0.5, γ=0.01, δ=0.1, μ=2, f=ka=0.1, kh=π, kr=0.5]

coexistence of nonlinear solutions and period doubling bifurcations may cascade resulting in chaotic response.

The bias induced by the quadratic drag force is the mechanism which generates the secondary resonance consisting of the period doubled response even for moderate sea states. The 1/2 subharmonic motion predicted cannot be obtained from a model in which the hydrodynamic exciting moment is equivalently linearized. Note that the period doubled response is of greater magnitude than the response predicted by the equivalent system.

Thus, stability analysis of this nonlinear ocean system can predict the complex dynamics recently uncovered numerically.

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APPENDIX

A.1 Amplitude equations (Eq. 5):

$G_i(A_0, A_m, \Phi_m)=0$

where $I=3, M=1$: $\theta = A_0 + A_1 \cos(\omega t+\Phi_1)$

or

$$R_0^2+\frac{1}{2}\left(R_{1S}^2+R_{1C}^2+R_{2C}^2+R_{3C}^2\right)-\frac{(\delta f)^2}{8}\left(2S_o^2+S_{2S}^2+S_{2C}^2\right)=0$$

$$2R_oR_{1S}+R_{2C}(R_{1C}+R_{3C})=0$$

$$2R_oR_{1C}+R_{2C}R_{1S}=0 \qquad \text{(A.1a)}$$

where

$$R_0=\alpha A_o\left[1-\frac{1}{6}\left(A_o^2+\frac{3}{2}A_1^2\right)\right]$$

$$R_{1S}=\mu\kappa_\mu f\cos\Phi_1+\gamma\omega A_1$$

$$R_{1C}=(\alpha-\omega^2)A_1-\mu\kappa_\mu f-\frac{1}{8}\alpha A_1(A_1^2+4A_o^2)$$

$$R_{2C}=\frac{1}{4}\alpha A_oA_1^2$$

$$R_{2S}=\frac{1}{24}\alpha A_1^3 \qquad \text{(A.1b)}$$

$$S_o=K_{\delta 1}f^2+2K_{\delta 2}fA_1\sin\Phi_1+K_{\delta 3}A_1^2$$

$$S_{2S}=K_{\delta 1}f^2+2K_{\delta 2}fA_1\cos\Phi_1$$

$$S_{2C}=K_{\delta 1}f^2\cos 2\Phi_1-2K_{\delta 2}fA_1\sin\Phi_1-K_{\delta 3}A_1^2 \qquad \text{(A.1c)}$$

and

$$K_{\delta 1}=\frac{1}{4kr}\left(\frac{2}{\sinh 2kh}+\frac{2}{kh}-\frac{\tanh kh}{(kh)^2}\right)$$

$$K_{\delta 2}=\frac{1}{kr}\left(\tanh kh-\frac{2}{kh}+\frac{2\tanh kh}{(kh)^2}\right)$$

$$K_{\delta 3}=\frac{1}{4kr}(kh)^2\tanh kh \qquad \text{(A.1d)}$$

A.2 Coefficients for the harmonic Hill's variational equation (Eq. 11):

$$\lambda_0=1-\frac{1}{2}\left(A_0^2+\frac{1}{2}A_1^2\right)$$

$$\lambda_1=-A_0A_1$$

$$\lambda_2=-\frac{1}{4}A_1^2 \qquad \text{(A.2a)}$$

$$\mu_{1C}=K_{\delta 2}f\cos\Phi_1$$

$$\mu_{1S}=K_{\delta 2}f\sin\Phi_1+K_{\delta 3}A_1 \qquad \text{(A.2b)}$$

A.3 Coefficients for the subharmonic Hill's variational equation (Eq. 16):

$$\lambda_0'=1-\frac{1}{2}\left[A_0^2+\frac{1}{2}\left(A_{1/2}^2+A_1^2\right)\right]$$

$$\lambda_{C/2}'=-\frac{1}{2}\left[2A_0A_{1/2}\cos\Phi_{1/2}+A_{1/2}A_1\cos\left(\Phi_1-\Phi_{1/2}\right)\right]$$

$$\lambda_{S/2}'=+\frac{1}{2}\left[2A_0A_{1/2}\sin\Phi_{1/2}+A_{1/2}A_1\sin\left(\Phi_1-\Phi_{1/2}\right)\right]$$

$$\lambda_{1C}'=-\frac{1}{2}\left[\frac{1}{2}A_1^2\cos(2\Phi_{1/2})+2A_0A_1\cos\left(\Phi_1\right)\right]$$

$$\lambda_{1S}'=+\frac{1}{2}\left[\frac{1}{2}A_1^2\sin(2\Phi_{1/2})+2A_0A_1\sin\left(\Phi_1\right)\right] \qquad \text{(A.3a)}$$

$$\mu_{S/2}'=\frac{1}{2}K_{\delta 3}A_{1/2}\cos\Phi_{1/2}$$

$$\mu_{C/2}'=\frac{1}{2}K_{\delta 3}A_{1/2}\sin\Phi_{1/2}$$

$$\mu_{1S}'=-K_{\delta 2}f\sin\Phi_1+K_{\delta 3}A_1\cos\Phi_1$$

$$\mu_{1C}'=-K_{\delta 2}f\cos\Phi_1+K_{\delta 3}A_1\sin\Phi_1 \qquad \text{(A.3b)}$$