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# Analysis of a Nonlinear System Exhibiting Chaotic, Noisy Chaotic, and Random Behaviors

*This study presents a stochastic approach for the analysis of nonchaotic, chaotic, random and nonchaotic, random and chaotic, and random dynamics of a nonlinear system. The analysis utilizes a Markov process approximation, direct numerical simulations, and a generalized stochastic Melnikov process. The Fokker-Planck equation along with a path integral solution procedure are developed and implemented to illustrate the evolution of probability density functions. Numerical integration is employed to simulate the noise effects on nonlinear responses. In regard to the presence of additive ideal white noise, the generalized stochastic Melnikov process is developed to identify the boundary for noisy chaos. A mathematical representation encompassing all possible dynamical responses is provided. Numerical results indicate that noisy chaos is a possible intermediate state between deterministic and random dynamics. A global picture of the system behavior is demonstrated via the transition of probability density function over its entire evolution. It is observed that the presence of external noise has significant effects over the transition between different response states and between co-existing attractors.*

## Introduction

The effects of noise on nonlinear dynamical systems exhibiting chaotic behavior have been of interest to researchers in various fields in recent years. Frey and Simiu (1992) derived a generalized stochastic Melnikov function by considering the additive noise as a perturbation to the homoclinic orbit, and concluded that the presence of weak noise can not suppress chaotic motion. Probabilistic properties of noisy nonlinear responses have been demonstrated by probability density functions. By solving the Fokker-Planck equation for marginal probability density functions of nonlinear systems driven by periodic excitation and random noise, Kapitaniak (1988) observed that noisy chaotic response is nonstationary and can be characterized by a multi-maxima curve in marginal probability density functions. Examining the noise effect on the response behavior of nonlinear systems through the Lyapunov exponent and probability density function, Bulsara et al. (1990) numerically demonstrated the noise-induced chaos and smoothing effect. Employing a finite difference procedure to solve the Fokker-Planck equation, Kunert and Pfeiffer (1991) found that the resulting steady-state joint probability density function can portray the corresponding chaotic attractor on the Poincaré section.

This investigation systematically analyzes all the possible dynamical responses in a nonlinear system from a stochastic perspective. The links among distinct (in a classical sense) dynamical responses (i.e., nonchaotic, chaotic, random and nonchaotic, random and chaotic, and purely random) in a nonlinear system are examined. By introducing a modulation factor which governs the relative strengths of deterministic and stochastic forcings, all possible dynamics can be formulated under a single mathematical representation. By varying this factor, diverse dynamical phenomena are exhibited, and possible transition routes from deterministic responses to purely random responses can be demonstrated. Probabilistic representation of noisy responses is provided by the probability density function, which is obtained by solving the Fokker-Planck equation. The path integral solution is employed to numerically solve the Fokker-Planck equation to demonstrate the evolution of probability density function (not examined in previous works), which characterizes the global system behavior. Moreover, the steady-state probability density functions can reflect existing response attractors. The noise-induced transition between the co-existing attractors is also examined. A generalized stochastic Melnikov process is develop to identify a boundary for possible chaotic domain by extending the deterministic Melnikov function to include stochastic excitations. Contrasted to the work by Frey and Simiu (1992), ideal white noise instead of Shinozuka's band-limited noise is used in our formulation. Noise-induced transition from nonchaotic response to noisy chaotic response becomes analytically evident when a mean-square representation of generalized stochastic Melnikov criterion is derived. Although the proposed analysis approach is completely general, for convenience of demonstration, the well-studied Duffing system is employed.

## System Considered

A periodically forced Duffing system with additive random perturbation is given by

$$\ddot{x} + c\dot{x} - x + x^3 = (1 - \gamma)B \cos \omega t + \gamma\xi(t) \tag{1}$$

where $\xi(t)$ is a zero-mean and delta-correlated Gaussian white noise with noise intensity $\nu$. In Eq. (1), $\gamma$ is a modulation factor determining the relative strengths of deterministic and stochastic forcings. By varying $\gamma$, the full range from purely deterministic systems ($\gamma = 0$) to purely random systems ($\gamma = 1$) is represented. In the intermediate states ($0 < \gamma < 1$), the Duffing system is subjected to both periodic excitation and Gaussian white noise. The noise effect on the nonlinear response can be examined by fixing the deterministic excitation and varying the noise intensity. For this purpose, the modulated periodic forcing amplitude, $A = (1 - \gamma)B$, and noise intensity, $\kappa = \gamma^2\nu$, are introduced. Equation (1) becomes

$$\ddot{x} + c\dot{x} - x + x^3 = A \cos \omega t + \eta(t) \tag{2}$$

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where $\eta(t)$ is a Gaussian white noise with modulated intensity $\kappa$. Depending on the purpose of illustration, Eqs. (1) and (2) are utilized alternatively. The corresponding Hamiltonian (unperturbed) system can be obtained by introducing two state variables, $q$ and $p$:

$$\begin{Bmatrix} \dot{q} \\ \dot{p} \end{Bmatrix} = f(q, p) = \begin{Bmatrix} \dfrac{\partial H(q, p)}{\partial p} \\ -\dfrac{\partial H(q, p)}{\partial q} \end{Bmatrix} = \begin{Bmatrix} p \\ q - q^3 \end{Bmatrix} \quad (3)$$

where $[q, p]^T = [x, \dot{x}]^T$, and $H(q, p)$ represents the Hamiltonian (Guckenheimer and Holmes, 1983). Equation (2) can be expressed as follows:

$$\begin{Bmatrix} \dot{q} \\ \dot{p} \end{Bmatrix} = f(q, p) + g(q, p; t);$$

$$g(q, p; t) = \begin{Bmatrix} 0 \\ -cp + A\cos\omega t + \eta(t) \end{Bmatrix} \quad (4)$$

where $g(q, p; t)$ is the perturbation to the Hamiltonian system.

## Methods of Analysis

To examine the (noisy) nonlinear behavior of the system in a stochastic sense, a Markov process approach is described in detail. Generalized stochastic Melnikov process is developed to analytically demonstrate noise-induced transition. Direct numerical simulations are employed to validate the analytical predictions.

**Markov Process Approach.** The behavior of a nonlinear system subjected to periodic excitation and white noise can be approximated as a Markov process (Kapitaniak, 1988), and the conditional probability density function of this process is determined by the knowledge of the most recent condition (Risken, 1984). Probability density functions of a Markov process obeys a statistically equivalent deterministic partial differential equation—the Fokker-Planck equation. The temporal solution to this equation can be obtained using a path integral solution procedure.

*Fokker-Planck Equation.* The Fokker-Planck equation associated to Eq. (3) is given by

$$\frac{\partial \mathbf{P}(q, p, t)}{\partial t} = -\frac{\partial}{\partial q}\{p\mathbf{P}(q, p, t)\} - \frac{\partial}{\partial p}\{(-cp + q - q^3 + A\cos\omega t)\mathbf{P}(q, p, t)\} + \frac{\kappa}{2}\frac{\partial^2}{\partial p^2}\mathbf{P}(q, p, t) \quad (5)$$

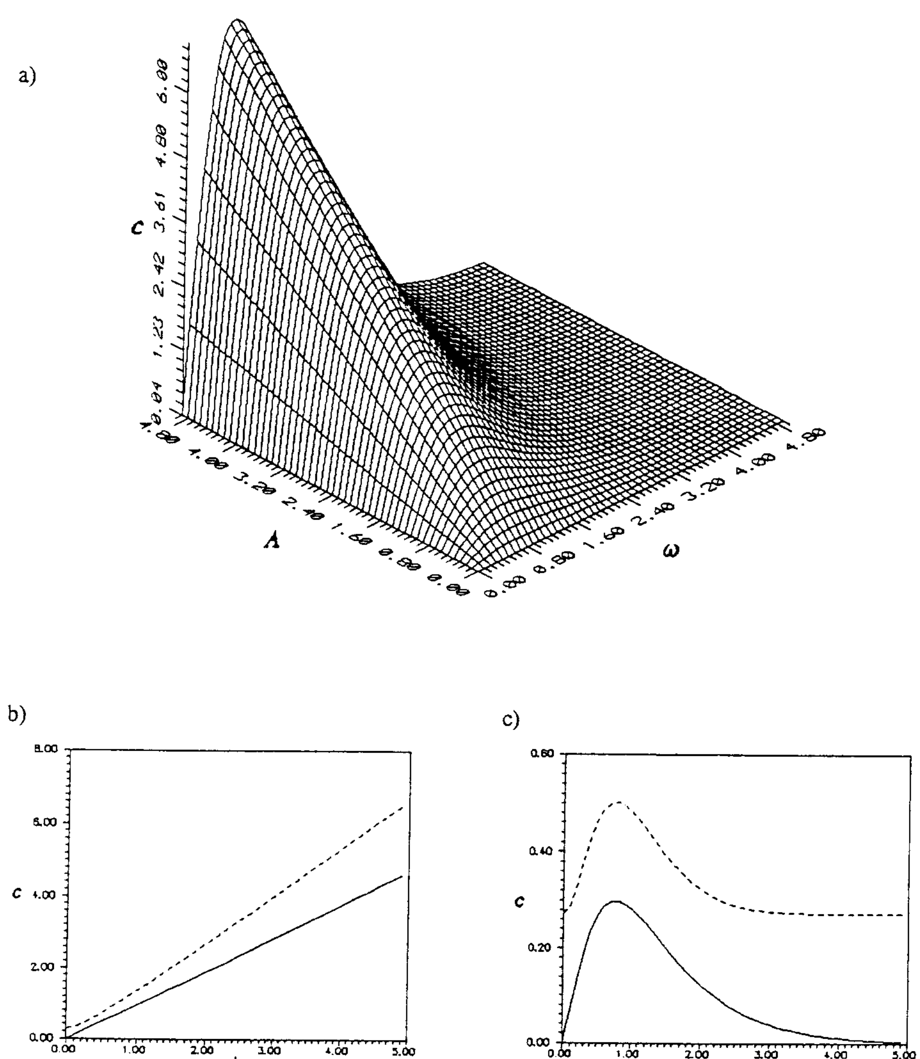


Fig. 1 Upper bound for possible chaotic domain based on generalized stochastic Melnikov process: (*a*) upper surface in *c*-*A*-$\omega$ space; (*b*) upper bound in *c*-*A* plane with $\omega = 1.0$; (*c*) upper bound in *c*-$\omega$ plane with $A = 0.3$; $\kappa = 0.0$ and 0.01 for solid and dashed lines, respectively.

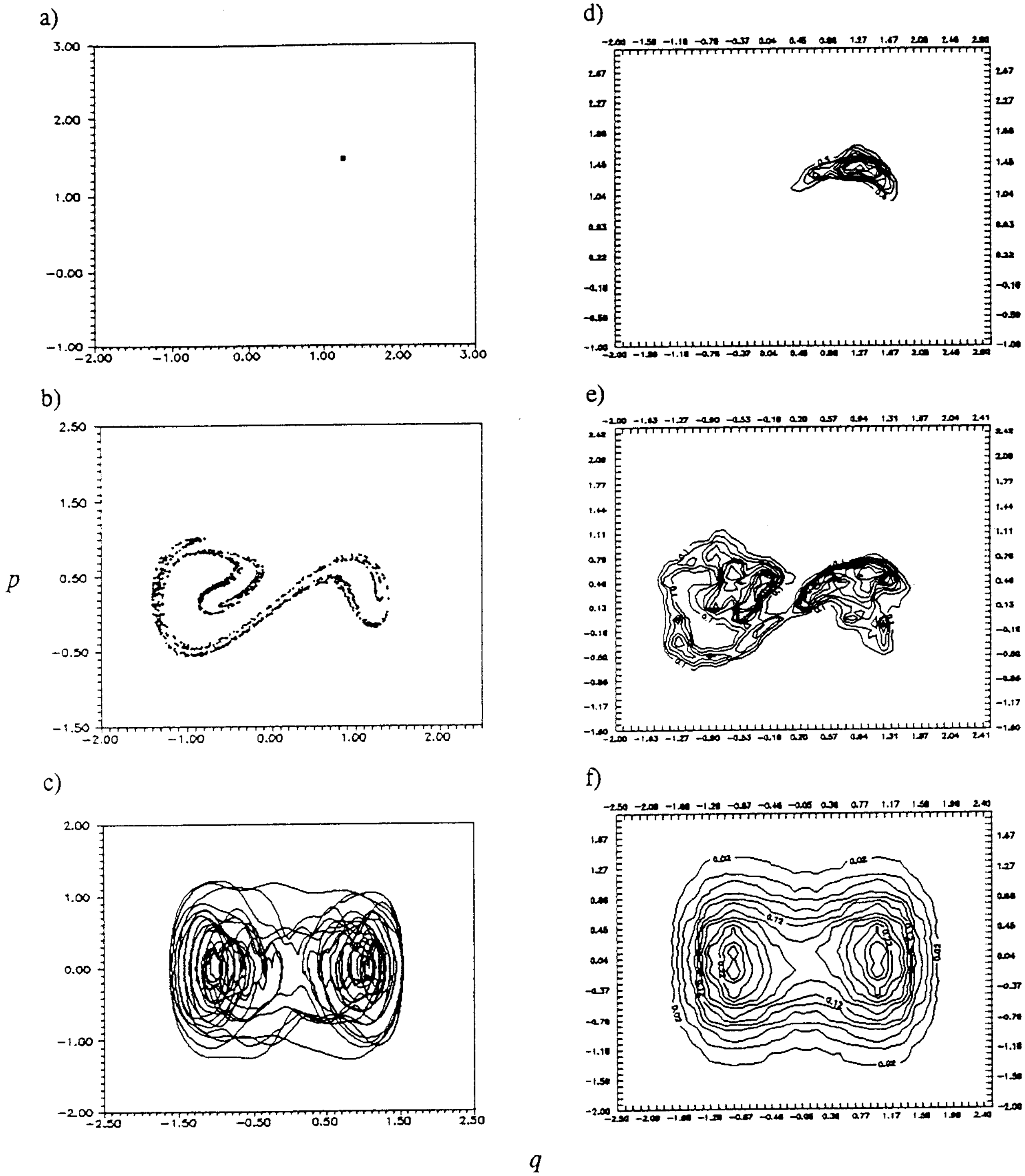


Fig. 2 Possible dynamical responses: (*a*) deterministic periodic attractor (Poincaré map); (*b*) deterministic chaotic attractor (Poincaré map); (*c*) sample realization of random response (phase plane); joint probability density functions of (*d*) random and periodic, (*e*) random and chaotic, and (*f*) random responses.

where $\mathbf{P}(q, p, t)$ denotes the probability density function, $p$ and $-cp + q - q^3 + A \cos \omega t$ are the two components in the drift vector, and $\kappa$ is the only nonzero entry in the diffusion matrix (Gardiner, 1985). The periodic excitation in the drift vector (Eq. (5)) implies that the probability density function is periodic with period $2\pi/\omega$ in time (Stratonovich, 1967). The periodicity in the joint probability density function can be suppressed by sampling the probability on the Poincaré section, $\mathbf{P}_p(q, p)$, and the steady state of which appears invariant (Kunert and Pfeiffer, 1991). This representation of probability density function will be used throughout the study.

*Path Integral Solution.* In the path integral solution procedure, the traveling path of the probability density function is discretized in terms of infinitesimal segments in probability space (Wissel, 1979). Each segment represents a short time propagation which is approximated by a time-dependent Gaussian distribution called the short time probability density

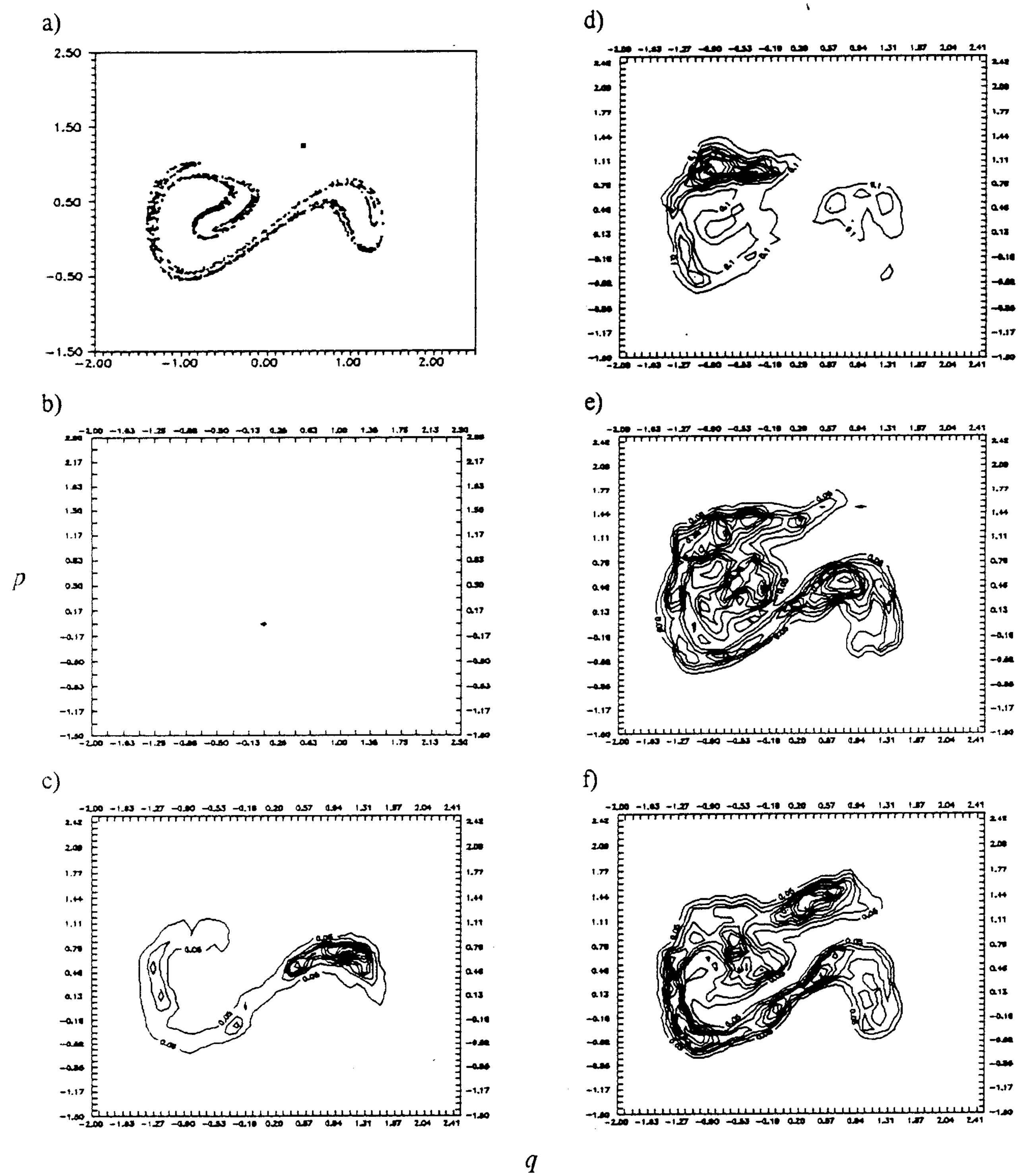


Fig. 3 Evolution of joint probability density function: (a) coexisting deterministic chaotic and periodic attractors in Poincaré map with $c = 0.163$, and initial conditions at (0, 0) and (1.0, 1.5), respectively; joint probability density function at (b) initiation, and at (c) the 2nd, (d) the 3rd, (e) the 5th, and (f) the 25th cycle of the forcing period, $c = 0.185$; $(A, \omega, \kappa) = (0.3, 1.0, 0.003)$.

function. The mean and variance of this probability density function are determined by the drift vector and the diffusion matrix, respectively. At the succeeding state, the probability density function can be obtained through a short time propagation and the probability density function at the desired state can be obtained by applying the propagation iteratively. For the system considered the short time probability density function, $G(q', p', q, p, t; \tau)$, is given by

$$G(q', p', q, p, t; \tau) = \frac{1}{\sqrt{2\pi\kappa\tau}} \exp\left[ -\frac{\tau}{2\kappa}\left( -cp + q - q^3 + A\cos\omega t - \frac{p'-p}{\tau} \right)^2 \right] \delta\left( p - \frac{q'-q}{\tau} \right) \quad (6)$$

where vectors $[q', p']^T$ and $[q, p]^T$ represent the post-state and pre-state, respectively, and the probability density function at the time $t$ is given by

$$\mathbf{P}(q,p,t) = \lim_{\substack{\tau\to 0\\ N\to\infty\\ N\tau\to t-t_o}} \prod_{i=0}^{N-1} \int \dots \int \exp[-\tau \sum_{j=0}^{N-1} \mathbf{G}(q_{i+1}, p_{i+1}, q_i, p_i, t_j; \tau)]\mathbf{P}(q_o, p_o, t_o)dq_i\, dp_i \quad (7)$$

where $\mathbf{P}(q_o, p_o, t_o)$ is the probability density function of the initial conditions. The path integral solution procedure converges to the exact solution in the limit as $N \to \infty$ and $\tau \to 0$. It is noted that, as $\kappa \to 0$, the path integral solution converges to the path of the deterministic (single) response trajectory. The numerical approach of the path integral is based on a discrete lattice representation (Wehner and Wolfer, 1983). By discretizing the probability domains at the ends of the segments (i.e., pre-state and post-state), the short-time probability density function is converted to the transition tensor, and the probability density function at the post-state is obtained by accumulating all contributions from the pre-state. The desired probability density function can be achieved by iterating the above procedure. Numerical results obtained in this study indicate that the path integral solution procedure is quite computationally efficient.

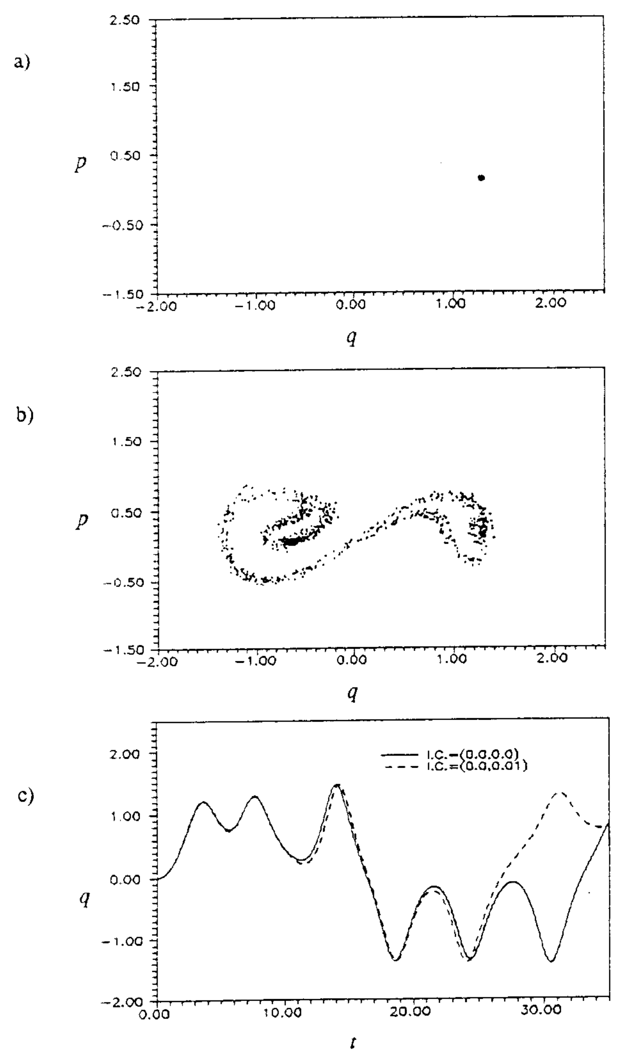


Fig. 4 Noise-induced chaos: (*a*) periodic response with $\gamma\sigma = 0.0$ (Poincaré map); (*b*) noise-induced chaotic response with $\gamma\sigma = 0.05$ (Poincaré map); (c) sensitivity to initial conditions (time history, corresponding to (*b*)); $(A, \omega, c) = (0.25, 1.0, 0.185)$.

**Generalized Melnikov Method.** A generalized version of the Melnikov function for a system subjected to an excitation with multiple frequencies was introduced by Wiggins (1988, 1990). The generalized stochastic Melnikov function (or more appropriately, the generalized stochastic Melnikov "process," due to its time dependency and random nature) for the Duffing system subjected to periodic and weak white noise is derived here. Unlike previous studies, a criterion based on the generalized Melnikov process is developed using a mean square representation to explicitly express the noise effects on possible occurrence of chaotic response.

The generalized stochastic Melnikov process is performed by treating the weak noise as a perturbation along with damping and periodic excitation. Assuming smallness of the perturbations, $cp$, $A\cos\omega t$, and $\eta(t)$, and expressing the pair of homoclinic orbits by explicit time functions, $(q^o(t), p^o(t)) = (\pm\sqrt{2}\,\text{sech}\,(t), \mp\sqrt{2}\,\sinh\,(t)\,\text{sech}^2(t))$ (Guckenheimer and Holmes, 1983), then the generalized stochastic Melnikov process is given by

$$\begin{aligned} M_g^+(t_{1o}, t_{2o}) &= \int_{-\infty}^{\infty} f[q_+^o(t), p_+^o(t)] \wedge g[q_+^o(t), p_+^o(t); t_{1o}, t_{2o}]dt \\ &= -\sqrt{2}\,A \int_{-\infty}^{\infty} \frac{\tanh(t)\cos\omega(t+t_{1o})}{\cosh(t)}dt \\ &\quad + 2c\int_{-\infty}^{\infty} \frac{\tanh^2(t)}{\cosh^2(t)}dt \\ &\quad - \sqrt{2}\int_{-\infty}^{\infty} \frac{\tanh(t)\eta(t+t_{2o})}{\cosh(t)}dt \\ &= M_d^+(t_{1o}) + M_r^+(t_{2o}) \end{aligned} \quad (8)$$

where "+" (superscript or subscript) indicates the upper domain of phase plane. The first two integrals in Eq. (8), corresponding to $M_d^+(t_{1o})$, represent the mean of the stochastic Melnikov process due to the periodic excitation and damping force, and the last integral, corresponding to $M_r^+(t_{2o})$, represents the random portion due to white noise. $M_r^+(t_{2o})$, instead of directly integrated, is calculated by considering the convolution integral as a filtering process, i.e., a white noise process $\eta(t)$ passing through a linear filter given by the homoclinic orbits $p_+^o(t)$ (Frey and Simiu, 1992). Because of the linear filtering, the random process $M_r^+(t_{2o})$ is stationary and of zero mean

$$\langle M_r^+(t_{2o})\rangle = \sqrt{2}\int_{-\infty}^{\infty} \frac{\tanh(t)}{\cosh(t)}\langle\eta(t_{2o})\rangle = 0. \quad (9)$$

Its corresponding spectrum is obtained by directly multiplying white noise spectrum $S_\eta(\omega)$ and the transfer function $|F(\omega)|^2$ of the linear filter, $p_+^o(t)$. The variance of $M_r^+(t_{2o})$ can be calculated by integrating its spectrum over the entire frequency range

$$\sigma_{M_r}^2 = \int_{-\infty}^{\infty} |F(\omega)|^2 S_\eta(\omega)d\omega \simeq 13.15\kappa \quad (10)$$

The frequency response $F(\omega)$ is obtained through Fourier transform

$$F(\omega) = \int_{-\infty}^{+\infty} p_+^o(t)e^{-i\omega t}dt = \frac{-i\omega\pi}{\sqrt{2}\cosh\frac{\pi\omega}{2}} \quad (11)$$

The generalized stochastic Melnikov criterion is obtained by setting Eq. (8) equal to zero

$$M_g^+(t_{1o}, t_{2o}) = M_d^+(t_{1o}) + M_r^+(t_{2o}) = 0. \quad (12)$$

Note that the stochastic Melnikov process (Eq. (8)) is Gaussian

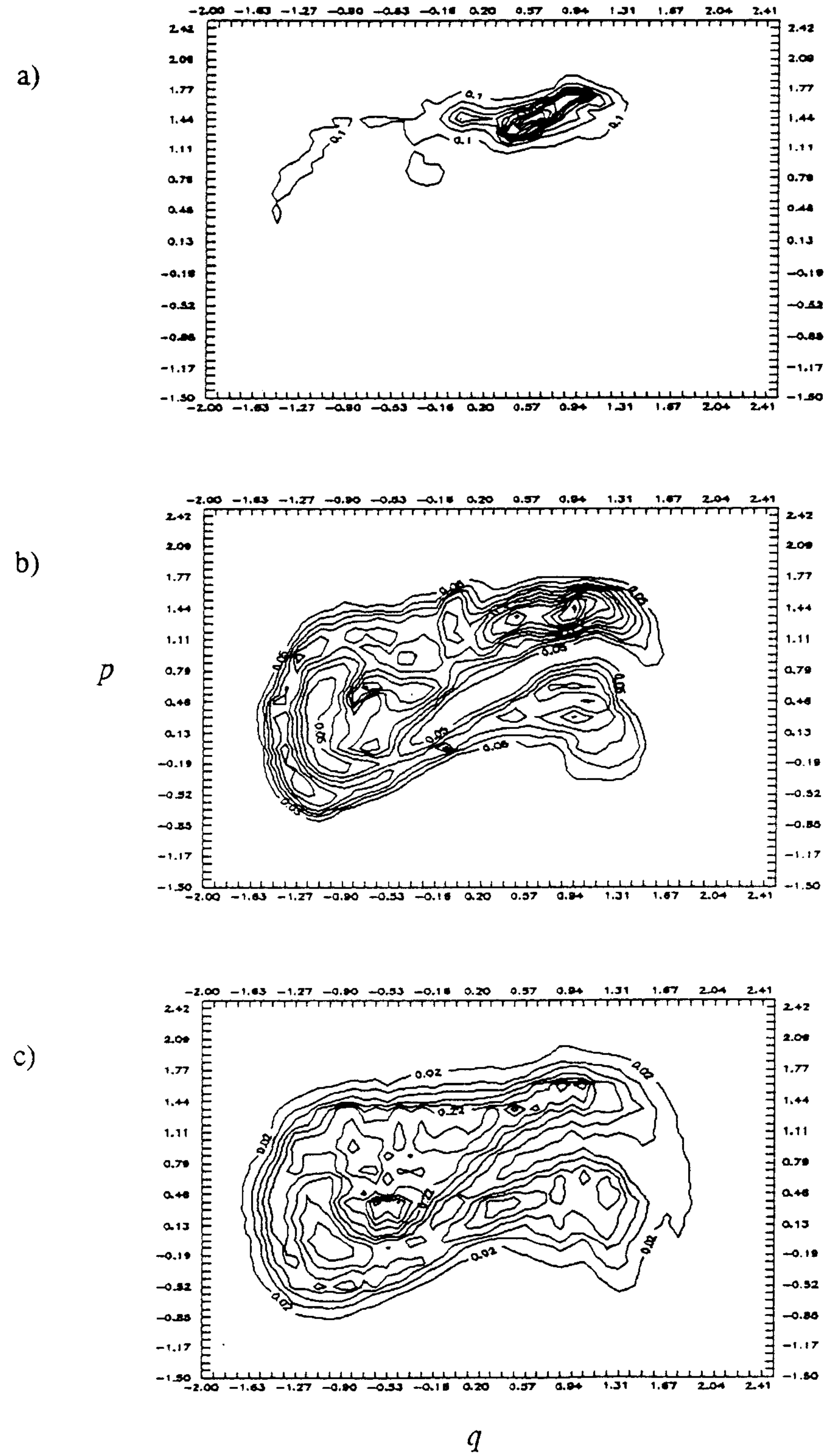


**Fig. 5 Transition between coexisting attractors caused by various noise intensities: (*a*) $\kappa$ = 0.001, (*b*) $\kappa$ = 0.007, and (*c*) $\kappa$ = 0.02, ($A$, $\omega$, $c$, $\kappa$) = (0.3, 1.0, 0.185, 0.003).**

with mean and variance equal to $M_d^+(t_{1o})$ and $\sigma^2_{M_r}$, respectively. In the limit as $\kappa \to 0$, the generalized stochastic and the standard deterministic Melnikov criteria coincide.

The standard Melnikov function renders a necessary condition for existence of chaos (Yim and Lin, 1991), and in this study the noise effects on the occurrence of possible chaotic response can be explicitly expressed by an energy interpretation. Thus, in regard to the presence of noise, the criterion for possible chaotic response based on generalized Melnikov process is performed in a mean-square representation

$$\left\langle \left(\frac{4c}{3}\right)^2 \right\rangle = \left\langle \left(\frac{\sqrt{2}\, A\pi\omega \cos\,(t_{10})}{\cosh \dfrac{\pi\omega}{2}}\right)^2 \right\rangle + \langle M_r^2(t_{2o})\rangle. \quad (13)$$

Then the mean-square criterion for possible chaotic response in terms of parameters ($A$, $\omega$, $c$, $\kappa$) is given by

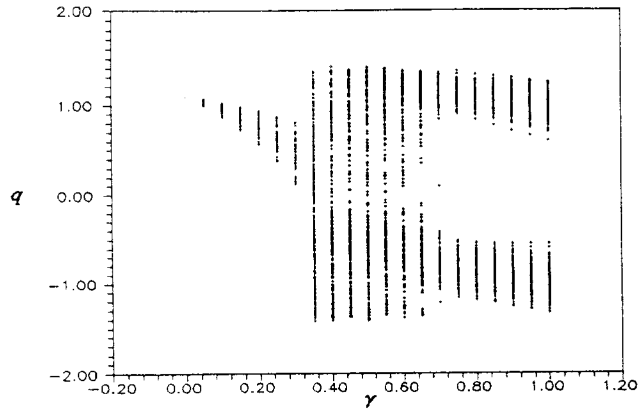


**Fig. 6 Transition among nonchaotic, noisy chaotic, and random dynamics (Poincaré points versus $\gamma$ with $(B, \omega, c, \sigma^2) = (0.5, 1.0, 0.185, 0.04)$).**

$$\left(\frac{4c}{3}\right)^2 \leq \frac{2A^2\pi^2\omega^2}{\cosh^2 \frac{\pi\omega}{2}} + \sigma_{M_r}^2. \quad (14)$$

The mean-square representation of the upper bound of possible chaotic domain is obtained by equating the expressions in Eq. (14) and portrayed in Fig. 1(*a*). It delineates the domain of possible occurrence of chaotic response in parameter space. The upper bounds for the cases with and without random perturbation in different parameter domains are also represented by the dashed and solid lines in Figs. 1(*b*) and (*c*), respectively. The positive correction term, $\sigma_{M_r}^2$ (Eq. (10)), elevates the upper bound for chaos (Figs. 1(*b*) and (*c*)). Thus, the presence of noise lowers the threshold and enlarges the possible chaotic domain in parameter space.

**Direct Numerical Simulation.** Numerical simulation is employed to validate the analytical predictions. Sample realizations of the system response are obtained by directly integrating Eq. (1), e.g., by a fourth-order Runge-Kutta integration algorithm. The numerical representation of the random noise selected here is based on Shinozuka's Gaussian noise (Shinozuka, 1977), the intensity of which is measured by its finite variance, $\sigma^2$. It is noted that Shinozuka's noise representation provides a good approximation to the ideal white noise (infinite variance with intensity scaled by the intensity parameter, $\kappa$), and it preserves the characteristics of the system response to ideal white noise excitation.

## Probabilistic Representation of Possible Responses

By varying the excitation parameters, the Duffing system exhibits diverse and complex dynamical responses including nonchaotic, chaotic, random and nonchaotic, random and chaotic, and purely random behaviors. Links among these possible responses are examined via transient and steady-state probability density functions obtained from the Markov process approach.

**Single Attractor.** Figures 2(*a*) and (*b*) show sample periodic and chaotic responses in the Poincaré map. By adding a small amount of noise ($\kappa = 0.003$) to the deterministic excitation, the corresponding joint probability density functions ($\mathbf{P}_p$) is obtained and shown on the Poincaré section in Figs. 2(*d*) and (*e*). It is observed that, with weak noise intensity, joint probability density functions of random and periodic (Fig. 2(*d*)), and random and chaotic responses can reflect the response attractors of their deterministic counterparts. It is noted that in Fig. 2(*d*) the fractal details in the chaotic attractor are clearly described by the joint probability density function.

Figure 2(*c*) demonstrates a sample realization of purely random response in the phase plane. It is observed that the phase trajectory drifts back and forth between the two basins in the potential. These two basins are reflected by the associated joint probability density function (Fig. 2(*f*)).

The above numerical results indicate that probability density functions not only can delineate purely random response (Fig. 2(*c*)) but also can reflect deterministic attractors, both nonchaotic and chaotic, in a probabilistic sense when the noise is weak. Thus, the probability density function may be a measure applicable to all possible dynamical responses (Lasota and Mackey, 1994). The asymptotic behaviors of the probability density functions as the noise intensity goes to zero will be the focus of future investigations.

**Coexisting Attractors.** An advantage of using probability density functions to demonstrate stochastic responses is further elaborated by examining coexisting attractors in this section. The coexistence of attractors is one of the characteristics of nonlinear systems. Figure 3(*a*) shows coexisting deterministic chaotic and periodic attractors in the Poincaré map (with initial conditions at (0, 0) and (1.5, 1.0), respectively). The joint probability density function, which describes the global behavior of the system, can be used to effectively portray these two coexisting attractors (Kunert and Pfeiffer, 1991) (Fig. 3(*f*)).

This probabilistic representation of the coexistence of attractors is demonstrated by initiating the slightly randomly perturbed system with deterministic initial conditions in the domain of the chaotic attractor and sampling recurrently on the Poincaré section to demonstrate the evolution of the joint probability density function. Figures 3(*b*), (*c*), and (*d*) show the evolution of the probability density function from its deterministic initial conditions at (0, 0) to the domain of the chaotic attractor The probability density function then drifts from the chaotic attractor to the periodic attractor after five cycles of the forcing period (Fig. 3(*e*)). Finally, at about 20 cycles, steady state is achieved and the joint probability density function clearly reflects the two coexisting attractors (Fig. 3(*f*)). The probability density function describes the trajectories and demonstrates the strengths of attractors in the mean sense. Under a detailed examination of the numerical results of Fig. 3(*f*), it is found that the probability density function is more concentrated in the domain of the periodic response, indicating that the periodic attractor is of the greater strength.

It is noted that the steady-state probability density function is independent of the initial conditions. In particular, although the coexisting attractors shown in Fig. 3(*f*) are obtained with initial conditions in the domain of the deterministic chaotic attractor, numerical results (not presented here due to space limitation) confirm that the same steady-state probability density function can be obtained by initiating the system within the domain of the deterministic periodic attractor. This is because the probability density function, although concentrated at the attractor, covers the entire phase space. Transitions between attractors are guaranteed by the presence of random perturbations. Hence, initial conditions become insignificant in describing the global behavior of stochastic systems, which is in contrast with the sensitivity to the initial conditions of a single deterministic chaotic trajectory. A slight difference in system parameters is noted in Figs. 3(*a*) and (*f*), which is due to a noise-induced shift of threshold of different response states (Just, 1989).

**Noise-Induced Transition.** The generalized Melnikov criterion analytically elaborates that the presence of noise enlarges the possible chaotic domain. The steady-state probability density functions in the previous section also indicate the noise-induced bridging effects on coexisting attractors. In this section noise-induced transitions between distinct response states and between coexisting attractors are numerically examined and interpreted by fixing the deterministic excitation and varying the noise intensity.

Analytical prediction of the generalized Melnikov criterion is corroborated by the phenomenon called noise-induced chaos (Bulsara et al., 1990). This random and chaotic state is induced from the neighboring nonchaotic state due to the presence of weak noise perturbation. If the noise intensity increases, the system response exhibits more random-like behavior. The transition between nonchaotic response, and random and chaotic response caused by the presence of noise is demonstrated in Figs. 4($a$) and ($b$). Figure 4($a$) shows a sample deterministic periodic response (no random perturbation) with quiescent initial conditions. Figure 4($b$) shows that, with the same initial conditions, a random and chaotic state is induced by a weak noise perturbation. Chaoticity of the response in Fig. 4($b$) can be illustrated by examining the sensitivity of its trajectory to the initial conditions. Figure 4($c$) shows that a slight variation in the initial conditions causes divergent response behavior and its chaoticity is demonstrated. As expected, with stronger noise intensity, the system response appears to become more random. Thus, with noise intensity as the control parameter in the vicinity of chaotic domain, random and chaotic response may be considered as an intermediate state between deterministic and random responses.

For coexisting periodic and chaotic attractors, numerical results indicate that the relative strength of the chaotic attractor is enhanced by the presence of a moderate amount of noise (Figs. 5($a$)–($c$)). Figure 5($a$) shows, with very low noise intensity, the probability density function is mostly concentrated in the domain of periodic response, and system response mainly exhibits periodic behavior. When the noise intensity increases, the probability density function covers the domains of both periodic and chaotic responses, and the system behaves with characteristics of both responses. When the noise intensity is of relatively high strength, domains of periodic and chaotic responses are merged, and system behavior appears more random. Thus the noise intensity should be considered as an important parameter in search of chaotic response in practice.

**Transition Between Deterministic and Random Dynamics.** Diverse transition routes from deterministic responses to purely random results may result when the modulation factor $\gamma$ in Eq. (1) varies from zero to one. Possible transition routes are 1. "non-chaotic" → "random and non-chaotic" → "random"; 2. "chaotic" → "random and chaotic" → "random"; 3. "non-chaotic" → "random and chaotic" → "random". Routes 1 and 2 are more easily understood, and route 3 is not as intuitively evident.

Route 3 is numerically demonstrated, with quiescent initial conditions, a possible transition from purely deterministic to purely random dynamics by varying $\gamma$ in this section (Fig. 6). For each value of $\gamma$, the Poincaré points of ten random sample paths of the steady-state responses are combined and presented in the map in Fig. 5. The system response behaves in a deterministic periodic state at $\gamma = 0$. By examining the Poincaré map of each individual sample path, it is found that as $\gamma$ increases, the system response behaves in a noisy periodic fashion up to $\gamma = 0.3$ (indicated by the closeness of the sample Poincaré points in Fig. 5). The typical sample response within the interval of $\gamma$ between [0.35, 0.65] deviates to, and stays in, a noisy chaotic state, which is indicated by the large scattering of the Poincaré points in that region. When $\gamma$ increases further to beyond 0.65, the typical response becomes more random and in the limit as $\gamma$ approaches 1, the distribution of the Poincaré points approaches that of the steady-state response of a (pure) randomly excited double-well Duffing system (Lin, 1967).

As demonstrated above, "random and chaotic" response may be a possible intermediate state for system response drifting from the deterministic to the purely random. This highly nonlinear random response may be considered as a possible link between deterministic and random dynamics.

## Concluding Remarks

This study presents a stochastic analysis of all currently known responses in a nonlinear system, and demonstrates the system behavior in a probabilistic sense. Several conclusions are summarized as follows:

1 "Random and chaotic" response has been demonstrated to be a possible intermediate state between deterministic and random responses using a simple, well-studied system. Stochastic characteristics of all possible dynamical behaviors have been demonstrated.

2 Based on generalized stochastic Melnikov process, an upper bound for possible chaotic domain has been identified. The presence of noise lowers the threshold and enlarges the possible chaotic domain.

3 The Fokker-Planck equation along with path integral solution provide a temporal solution of probability density function, which characterizes the global behavior of the system. The steady-state joint probability density function appears invariant on the Poincaré section, which reflects all the existing attractors.

4 The presence of noise decreases the order in the system response. It bridges domains of all the existing attractors and controls their relative stabilities. Sensitivity to initial conditions of the "random and chaotic response" is less significant in the representation by probability density function.

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