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Unified Analysis of Complex Nonlinear Motions via Densities

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Abstract. A unified approach of using densities to analyze both deterministic and stochastic complex responses including chaotic and random motions of nonlinear engineering systems is illustrated in this study. Motivations to examine deterministic nonlinear dynamical systems via densities are first discussed. Essential mathematical background and techniques pertinent to the analyses of both deterministic chaos and random chaotic processes are briefly summarized. Densities of nonlinear responses are computed by numerically solving the Fokker–Planck equation to examine stochastic properties of random chaotic responses. It is demonstrated that, by introducing random perturbations in an otherwise deterministic excitation, the existence of attractors can be efficiently and clearly depicted by the evolution of a unique probability density over the physical phase space. Two distinct asymptotic behaviors of densities: (i) invariance and (ii) sweeping, of complex motions and their relationship to response stabilities predicted by the Foguel Alternative Theorem are numerically demonstrated. Applications using the probability densities to compute reliability indices of an engineering system are demonstrated.

Keywords: Nonlinear system, Fokker–Planck equation, probability density, Foguel alternative theorem.

1. Introduction

Two recent trends in the design of structures make the use probability density as an engineering analysis and design tool indispensable. One is the intended revision of many existing design codes to a reliability-based format. Another is the increasing demand for safe performance of a new generation of structures in a highly nonlinear (possibly inelastic) range.

In the past, many design codes for structures were deterministic with safety factors selected on an ad hoc basis. However, with the maturing of reliability theory and applications, and the availability of an increasing amount of field data, many on-going efforts have been devoted to converting deterministic-based codes to probabilistic-based ones (e.g. partial safety factors in the API LRFD code for offshore structures). To compute the reliability indices of a structure, probability distributions (or densities) of the response processes are needed.

Concurrently, the need for economical design and construction in often increasingly hostile environments requires structures to be compliant and to operate frequently in highly nonlinear modes. The nonlinear response motions include subharmonic, superharmonic, ultrasubharmonic, quasi-periodic and, under certain conditions, chaotic responses. Of these complex motions, the characterization of chaotic response, because of its seemingly unpredictable nature, presents the most challenge to the design engineer. However, recent research has discovered that although chaotic motions under regular (periodic) excitations are deterministic, they possess stochastic properties that can be efficiently characterized via densities. Although these properties are stochastically stable, they are difficult to be employed in practical engineering applications due to the fact that the invariant sets of chaotic attractors have fractal dimensions thus with Lebesque measure zero in physical phase space. Because of this, it is not clear how classical stochastic analyses and reliability calculations can be directly applied to purely chaotic motions. However, excitations in the natural environment (e.g. aerospace and ocean) often contain intrinsic random components. Thus, in addition to a dominant periodic component, random perturbations are present and need to be included in modeling of the excitations and the response analyses. Random chaotic response behavior may then be included in structural design via a probabilistic approach. The goal of this paper is to, along with other prominent researchers in the field [1–5], take another step forward in this direction by demonstrating the potential unified analysis of both deterministic and stochastic responses of engineering systems using (probability) densities.

2. Background

Chaotic behavior in a periodically driven deterministic nonlinear system has been of great interest to researchers in engineering [6, 7]. Criteria for occurrence of chaotic responses in purely deterministic nonlinear systems have been developed [8–10]. Applications of global analysis techniques on stability studies of ship roll motion have been carried out by Thompson et al. [11] using basin boundaries erosion, Gottlieb and Yim [12] using perturbation and harmonic balance methods, and Falzarano et al. [13] using lobe dynamics to explain the complex behavior and 'unexpected capsizing'.

It is well known that chaotic responses are sensitive to small variations in initial conditions and system parameters [6–8]. This 'unstable' characteristic makes it difficult to take into account chaotic responses in the design of nonlinear structures using conventional deterministic methods. On the other hand, chaotic responses possess many 'stable' stochastic properties. In fact, under properly selected measure spaces, chaotic attractors possess invariant densities (measures) and ergodic (and mixing) properties. These stable properties are difficult to employ in practical engineering applications because, as mentioned above, the invariant sets of chaotic attractors have fractal dimensions and Lebesque measure zero in physical phase space, and reliability indices of system response cannot be directly computed by applying conventional stochastic analyses and engineering reliability calculations.

However, under realistic field environments, purely deterministic (periodic) environmental excitations seldom exist and random perturbations are often inevitable. Thus the structural systems are better modeled as randomly perturbed. These so called noisy dynamical systems have been studied and it is concluded that deterministic analysis techniques via topological concepts may not be useful and global behavior including bifurcation phenomenon should be studied from a stochastic perspective [1–3, 14].

Stochastic extensions of a few analysis techniques originally developed for deterministic chaotic systems have been derived. In particular, the effects of weak random perturbations on chaotic behavior have been investigated via the Melnikov process and phase-space flux approaches [15–19].

Results of computer experiments on deterministic chaotic systems have suggested a wealth of stochastic phenomena of complex nonlinear system behaviors. Common stochastic properties among deterministic and randomly perturbed systems including existence of invariant densities (and measures), stochastic stability and asymptotic periodicity have been observed. Thus, stochastic concepts and ergodic theory can be applied to chaotic systems [1–5]. To gain an understanding of the relationships between deterministic systems exhibiting complex nonlinear behaviors and stochastic systems, knowledge of the basic stochastic calculus, ergodic and operator theories is helpful. A few key elements pertinent to interests in this study are briefly summarized below.

2.1. Mathematical Background

In order to introduce the mathematical concept of invariant measure, Markov and Frobenius–Perron operators and the Foguel Alternative Theorem, and their relationship with probability applications that are well-known to engineers, the notions of measure, density and operators pertinent to this study are first reviewed (see [20] for detailed descriptions).

μ is a *measure* of a space $(X, \mathcal{F})$ with $\mathcal{F}$ being a σ-algebra of subsets of a set X, and $\mathcal{F}$ satisfying: (i) $\mu(\phi) = 0$; (ii) $\mu(B) \geq 0$ for all $B \in \mathcal{F}$, and (iii) $\mu(\cup_k B_k) = \Sigma_k \mu(B_k)$ with $\{B_k\}$ finite or infinite sequence of pair-wise disjoint sets from $\mathcal{F}$. The triple $(X, \mathcal{F}, \mu)$ forms a measure space.

A function $f \in D(X, \mathcal{F}, \mu)$ is called a *density* if the set $D(X, \mathcal{F}, \mu) = \{g \in L^1(X, \mathcal{F}, \mu); g \geq 0 \text{ and } ||g|| = 1\}$ where L^1 is a Lebesque space consisting of all possible integrable functions $f : X \to \mathbb{R}$. Moreover, if $f \in L^1(X, \mathcal{F}, \mu)$ and $f \geq 0$, then the measure μ_f defined by

$$\mu_f(B) = \int_B f(x)\mu(\mathrm{d}x) \tag{1}$$

is absolutely continuous with respect to μ. In the special case that $f \in D(X, \mathcal{F}, \mu)$, then f is the density of μ_f and that μ_f is a *normalized measure*. Note that the probability theory employed in conventional reliability analyses is an application of measure theory with probability as a normalized measure.

The *Markov operator* is a primary tool in studying the flow of (probability) densities of stochastic systems. An operator $P_t : L^1 \to L^1$ in measure space $(X, \mathcal{F}, \mu)$ is called Markov with parameter t if $\forall f \geq 0$ and $f \in L^1$, and satisfies

$$P_t f \geq 0 \tag{2a}$$

and

$$||P_t f|| = ||f||. \tag{2b}$$

The Frobenius–Perron operator, which may be considered a deterministic restriction of the Markov operator, is useful for examining the flow of densities of corresponding deterministic systems. The part of ergodic theory concerning asymptotic behaviors of densities often applies equally well under both deterministic and stochastic settings. In particular, many analytical results applicable to deterministic flows described by Frobenius–Perron operators concerning chaotic behaviors evolving under the influence of periodic excitations have direct extensions to their corresponding stochastic counterparts under periodic excitations with random perturbations. The definition of Frobenius–Perron operators is introduced in the following.

A continuous *semigroup of operators* $S_t : X \to X, t \in \mathbb{R}^+$ may be defined in measure space $(X, \mathcal{F}, \mu)$ as the solution to the following ordinary differential equation [20]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b(x) \quad \text{with} \quad x(0) = x^0. \tag{3a}$$

A *Frobenius–Perron* operator can then be defined as the operator P_t satisfying

$$\int_B P_t f(x)\mu(\mathrm{d}x) = \int_{S_t^{-1}(B)} f(x)\mu(\mathrm{d}x), \tag{3b}$$

where $f \in D$ is a density associated with S_t

$$\int_{\mathbb{R}^m} f(x)\mu(\mathrm{d}x) = 1. \tag{3c}$$

These abstract concepts and definitions will assume concrete meaning when applied to the analysis of numerical examples.

2.2. Stochastic Calculus and Fokker–Planck Equation

2.2.1. *Fokker–Planck Equation*

The concept of the Markov operator has been applied in the classical nonlinear random vibration analysis (e.g., [21]). The response of nonlinear systems to white noise or filtered-white noise excitations are often modeled as a Markov process, and the flow of response probability densities is governed by the Markov operator. The evolution of the probability densities can be alternatively expressed by a statistically equivalent, deterministic partial deferential equation, called the Fokker–Planck equation (FPE). A generic form of the PFE is given by

$$\frac{\partial P}{\partial t} = \frac{1}{2}\sum_{ij=1}^{d} \frac{\partial}{\partial x_i}(b_i P) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i P) \tag{4}$$

for $t > 0, x \in \mathbb{R}^+$, with functions $\partial P/\partial t$, $\partial P/\partial x_i$, $\partial^2 P/\partial x_i \partial x_j$, $\partial a_{ij}/\partial x_i \partial x_j$, $\partial b_i/\partial x_i$, being continuous, and b_i, a_{ij} and their derivatives being bounded. Probability density $P(X, t)$ is a solution to the FPE if it satisfies

$$P(X, t) = \int_{\mathbb{R}^m} \Gamma(X, Y, t) f(Y)\,\mathrm{d}Y, \tag{5}$$

where $f(Y) \in L^1$ represents the initial distribution and $\Gamma(X, Y, t)$ is a stochastic kernel satisfying the corresponding FPE for $t > 0$. An advantage to this approach is that there is no limit to the degree of nonlinearity in the system considered.

This study focuses on applications of a few recently developed analysis techniques based on ergodic theory developed for deterministic systems and its stochastic extensions using densities to extract stability characteristics of randomly perturbed nonlinear complex (chaotic) motions. Based on the Frobenius–Perron operator theory, advantages of using evolution of (probability) density instead of single trajectory to illustrate the pertinent stochastic properties of chaotic response are numerically demonstrated. Random components in the excitation are later introduced to better describe the stochastic nature in the environmental loadings, e.g., wind, waves and current. Transient and steady-state probability densities are computed by numerically solving the (FPE), and their asymptotic behaviors are interpreted in light of the Foguel Alternative Theorem. The alternative behaviors: (i) invariant (stable), and (ii) sweeping (unstable), are numerically demonstrated. The key is to take advantage of the fact that the presence of a random excitation component induces stochastic responses of non-zero Lebesque measure in the state space which can be directly applied in computing engineering reliability indices.

3. System Model

To demonstrate the mathematical theory and engineering applications, a single-degree-of-freedom ship-roll model with water-on-deck effect [18], which possesses two distinctive well-defined response behavior domains, is considered here. Each response domain is examined individually to show the distinct characteristics of the associated response behavior with and without random perturbations.

Assuming a beam sea condition and the roll motion being uncoupled from other degrees of freedom, the governing equation of a single-degree-of-freedom ship-roll motion can be expressed as follows [18]

$$[I + A(\omega)]\ddot{\varphi} + B(\omega)\dot{\varphi} + B_q(\omega)\dot{\varphi}|\dot{\varphi}| + \Delta GZ_m(\varphi) = F_{\text{sea}}(\omega)\cos(\omega t + \varepsilon) + \xi(t), \tag{6a}$$

where φ is the roll angle, I is the moment of inertia (in air) of the ship about the roll axis, A is the hydrodynamic added mass coefficient, B is the linear roll damping coefficient, B_q is the quadratic drag coefficient, Δ is the weight of the ship, and $GZ_m(\varphi)$ is a polynomial approximation to the nonlinear roll-restoring moment arm. F_{sea}, ω and ε are the amplitude, frequency, and phase shift of the external wave exciting force, respectively. $\xi(t)$ is an ideal, zero-mean, delta-correlated Gaussian white noise, i.e.,

$$\begin{aligned} \langle \xi(t) \rangle &= 0, \\ \langle \xi(t')\xi(t) \rangle &= \nu\delta(t' - t), \end{aligned} \tag{6b}$$

where $\langle\cdot\rangle$ represents the ensemble average, $\delta(\cdot)$. Is the Dirac delta function, and ν is the noise intensity. The Gaussian white noise approximates all possible random perturbations in the external exciting force.

By including the effect of water-on-deck, the roll motion is characterized by two distinct, well-defined domains of different dynamical behaviors – homoclinic and heteroclinic [18]. Assuming frequency-independent coefficients, normalizing Equation (6a) with the inertia coefficient (mass plus added mass), linearizing the quadratic hydrodynamic force, taking a two-term polynomial approximation of the $GZ_m(\varphi)$ curve, and re-scaling time, the non-dimensionalized versions of the homoclinic and heteroclinic dynamics can be obtained, respectively, as follows

$$\ddot{x} + c\dot{x} - k_1 x + k_3 x^3 = A\cos(\omega t + \Psi) + \eta(t) \tag{7a}$$

and

$$\ddot{x} + c\dot{x} + k_1 x - k_3 x^3 = A\cos(\omega t + \Psi) + \eta(t), \tag{7b}$$

where $\eta(t)$ is a zero-mean, delta-correlated white noise with intensity κ, i.e.,

$$\begin{aligned} \langle \eta(t) \rangle &= 0 \\ \langle \eta(t')\eta(t) \rangle &= \kappa\delta(t' - t). \end{aligned} \tag{7c}$$

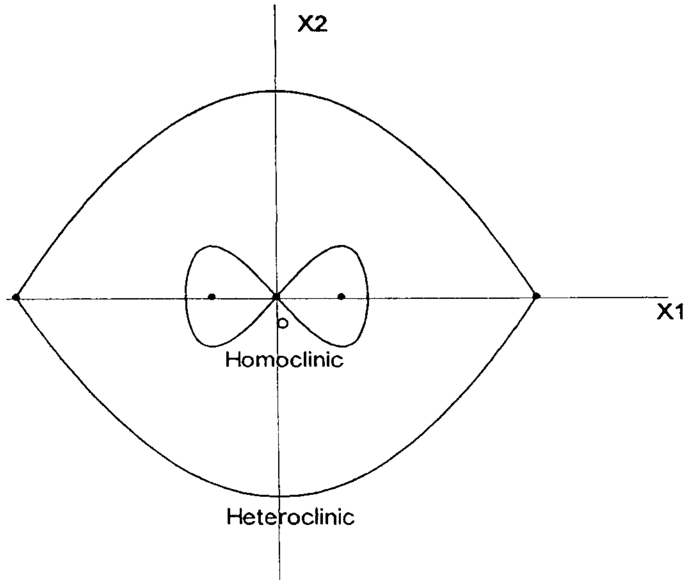

Figure 1. Homoclinic (inside) and heteroclinic orbits in unperturbed rolling motion.

The two systems (Equations (6a) and (6b)) can be rewritten via state variables, x_1 and x_2, and analyzed in the (displacement-velocity) state space

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ \mp k_1 x_1 \pm k_3 x_1^3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -cx_2 + A\cos(\omega t + \Psi) + \eta(t) \end{Bmatrix}, \tag{8a}$$

respectively, with

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}. \tag{8b}$$

The corresponding separatrix of each region in the phase space is plotted in Figure 1. Chaotic responses are found to exist near the separatrix in both regions [18]. The response behavior in each region with and without random perturbations will be examined and interpreted via the evolution and characteristics of (probability) densities in later sections.

The associated FPEs to Equations (7a) and (7b) may be written as [22]

$$\begin{aligned} \frac{\partial P(X,t)}{\partial t} &= -\frac{\partial}{\partial x_1}\{x_2\, P(X,t)\} - \frac{\partial}{\partial x_2}\{[-cx_2 + k_1 x_1 - k_3 x_1^3 \\ &\quad + A\cos(\omega t + \Psi)]P(X,t)\} + \frac{\kappa}{2}\frac{\partial^2}{\partial x_2^2} P(X,t) \end{aligned} \tag{9a}$$

and

$$\begin{aligned} \frac{\partial P(X,t)}{\partial t} &= -\frac{\partial}{\partial x_1}\{x_2\, P(X,t)\} - \frac{\partial}{\partial x_2}\{[-cx_2 - k_1 x_1 + k_3 x_1^3 \\ &\quad + A\cos(\omega t + \Psi)]P(X,t)\} + \frac{\kappa}{2}\frac{\partial^2}{\partial x_2^2} P(X,t), \end{aligned} \tag{9b}$$

where $P(X, t)$ denotes the (joint) probability density, x_2 and $(-cx_2 \pm k_l x_l \mp k_3 x_l^3 + A\cos(\omega t + \Psi))$ are the two entries in the corresponding drift vectors, and κ is the only non-zero coefficient in the 2×2 diffusion matrix [23]. The periodic excitation in the drift vector (Equation (9)) implies that the steady-state probability density, if exists, is periodic with period $2\pi/\omega$ in time [24, 25].

Although a simple analytical expression of the solution to the FPE (e.g. [9]), closed form solutions are seldom available in general. Among existing numerical schemes [1, 3, 26, 27], a path-integral solution procedure solving the FPE for probability densities is chosen here for its systematic numerical implementation and computational efficiency [27–29]. Computational advantages and limitations of the path-integral method have been discussed in detail in [5, 28, 29].

4. Deterministic and Random Chaotic Responses

4.1. Deterministic Chaotic Motions

Deterministic chaotic motions of a nonlinear system are often studied via modern nonlinear dynamical system approach [9–13]. Stability and global bifurcation behaviors of such a system can be illustrated by examining Equations (2a) and (b) with noise intensity κ zero. Possible deterministic chaotic responses are identified to exist when the stable and unstable manifolds transversely intersect each other [9, 10]. The intersections are estimated by applying the Melnikov function [8–10], and the resulting Melnikov criterion provides a lower bound for the chaotic domain in the parameter space. Deterministic chaotic responses of both the homoclinic and heteroclinic regions satisfying the Melnikov criterion are shown in Figure 2 to illustrate their non-periodicity in time history and phase plane, and the fractal properties in Poincaré map.

4.2. Randomly Perturbed Chaotic Motions

Random perturbations in the external excitation can be represented by a Gaussian white noise $\eta(t)$ with intensity 'modulation parameter' κ (Equation (7c)). The deterministic Melnikov criterion can be directly extended to take into account the presence of random perturbations. Specifically, with the presence of a zero-mean white noise $\eta(t)$ (Equation (7c)), a generalized (Gaussian distributed) stochastic Melnikov process and its variance can be obtained through the transfer function associated with the convolution integrals [18]. It is indicated that (in the mean-square sense) the presence of zero-mean random noise enlarges the possible chaotic domain in parameter space. Noise-induced effects on chaotic responses are numerically demonstrated by a single response trajectory represented via time history, phase plane and Poincaré map (Figure 3). Different stabilities of chaotic responses of the homoclinic and heteroclinic regions are indicated. It appears that under random perturbations the chaotic response in the homoclinic region is stable (Figures 3a–3c), and that in the heteroclinic region typically diverges (for the particular example considered, to lower left corner in phase plane and Poincaré map, Figures 3d–3f). Asymptotic stability of chaotic response of these two regions under random perturbations will be examined from an ensemble point of view via (probability) densities in the following sections.

5. Asymptotic Properties of Densities

5.1. Stochastic Properties of Deterministic Transformations

There are three levels of irregular response (chaotic) behaviors a deterministic (measure-preserving) transformation of nonlinear systems can induce – ergodicity, mixing, and exactness. The Frobenius–Perron operator is an efficient means to discern these behaviors [20]. However, the measure spaces suitable for analyzing these deterministic attractors and their stochastic properties, due to their fractal nature (for chaotic attractors), are usually different for each (possibly co-existing) attractor and have Lebesque measure zero associated with the physical state space of practical engineering interest. This undesirable (measure zero) property renders a direct application of the stochastic analyses of the deterministic attractors impractical for conventional reliability analyses.

5.2. Relationships between Deterministic and Randomly Perturbed Systems

As mentioned earlier, in the presence of random perturbations, the response of a nonlinear system is described by a stochastic differential equation. The corresponding response behaviors can be examined via the evolution of the density governed by the FPE. Solutions of the FPE are equivalent to the flows of densities governed by a semigroup of Markov operators.

When the relative intensity of the random perturbations in the excitation reduces to zero, the FPE reduces to the Liouville equation and the semigroup of Markov operators reduces to a semigroup of Frobenius–Perron operators. Under certain smoothness conditions (topological structural stability), properties associated with the deterministic system can be recovered as limiting cases of stochastic results.

Two major advantages of including random perturbations as a component of the excitation are that the model represents a more accurate description of the physical system, and that the resulting densities governing the responses belong to the physical state space of engineering interest. In fact the resulting densities are the familiar ones studied extensively in classical random vibrations, hence can be used directly in reliability calculations. Thus it is advantageous to examine nonlinear complex (chaotic) response behaviors from a stochastic setting.

5.3. Asymptotic Stability of Densities

While solutions to the FPE can be computed efficiently via numerical (path-integral solution) procedures, stability characteristics of the responses may be proficiently extracted by examining properties of their associated semigroup of Markov operators. Depending on the stability of the density, two distinct discrete asymptotic properties, (i) invariant and (ii) sweeping, can be observed. These two possible asymptotic behaviors of the density are addressed in the *Foguel Alternative Theorem*, which states that: (a) a continuous stochastic semigroup of Markov operators possesses either an asymptotic stationary density or sweeping properties for the FPE; and (b) if there exists a time shift t_0 such that the densities corresponding to t and $t + t_0$ are identical, then the semigroup of Markov operators is periodic and a stationary density which is equal to the averaged value of the densities over the period t_0 exists. Part (b) of the theorem can be expressed mathematically as follows: if there exists $t_0 \in \mathbb{R}^+$, $f_0 \in D$ t_0 such that

$$P_{t_0} f_0 = f_0, \tag{10a}$$

then

$$f_* = \frac{1}{t_0} \int_0^{t_0} P_t f_0(x)\, \mathrm{d}t \tag{10b}$$

and

$$P_t f_* = f_*, \quad \forall t \in \mathbb{R}^+. \tag{10c}$$

The mathematical conditions under which the theorem holds are described in detail in [15]. Note that despite the Foguel Alternative Theorem being demonstrated only by means of semigroup of Markov operators in this study, it is equally applicable to deterministic transformations, e.g., semigroup of Frobenius–Perron operators.

6. Numerical Results and Discussions

In the previous sections, essential mathematical background and theorems pertinent to engineer interest have been introduced. As many mathematical theorems, systematic procedures to verify particular engineering systems satisfying the mathematical conditions of a theorem are often difficult to develop and implement. However, the availability of such a mathematical framework and theorems are often helpful in interpreting numerical results. Possible engineering applications by using probability densities to incorporate intrinsic characteristics of both deterministic and randomly perturbed chaotic responses are indicated and discussed via numerical examples, and demonstrated in a later section. The usefulness of the Foguel Alternative Theorem to both deterministic and stochastic response analysis is also numerically demonstrated to illustrate periodicity and asymptotic stability as well as sweeping properties of the systems considered in this section.

6.1. Deterministic Chaos

Sample chaotic responses from both homoclinic and heteroclinic regions (Equations (7a) and (7b)) without random perturbations are first examined from a single trajectory and later from a stochastic perspective via probability densities.

6.1.1. *Trajectory vs. Density*

As demonstrated in Figure 2, chaotic responses are non-periodic in time, and their long-term unpredictability and sensitivity to uncertainties in initial conditions are further illustrated in Figure 4a. With slight variation from the initial conditions $(0, 0)$ and $(0, 0.1)$, respectively, dramatic changes are observed in the corresponding time histories (Figure 4a). Despite the unstable and unpredictable nature of the deterministic chaotic response, invariant stochastic properties can be extracted and depicted via density (Figure 4b).

Based on the theory of Frobenius–Perron operator (Section 2.1), ensemble numerical experiments are conducted to demonstrate stable properties of deterministic chaos with a uniform initial distribution. In this study, 5000 realizations of chaotic response with initial distribution corresponding to grid points in a tiny displacement-velocity square (0.001×0.001) centered around the initial condition of interest are generated and sampled after 500 cycles of the forcing period (Poincaré section). The chaotic attractor is clearly reflected by the contour plot of the density, which is invariant on the Poincaré section. Numerical results suggest that

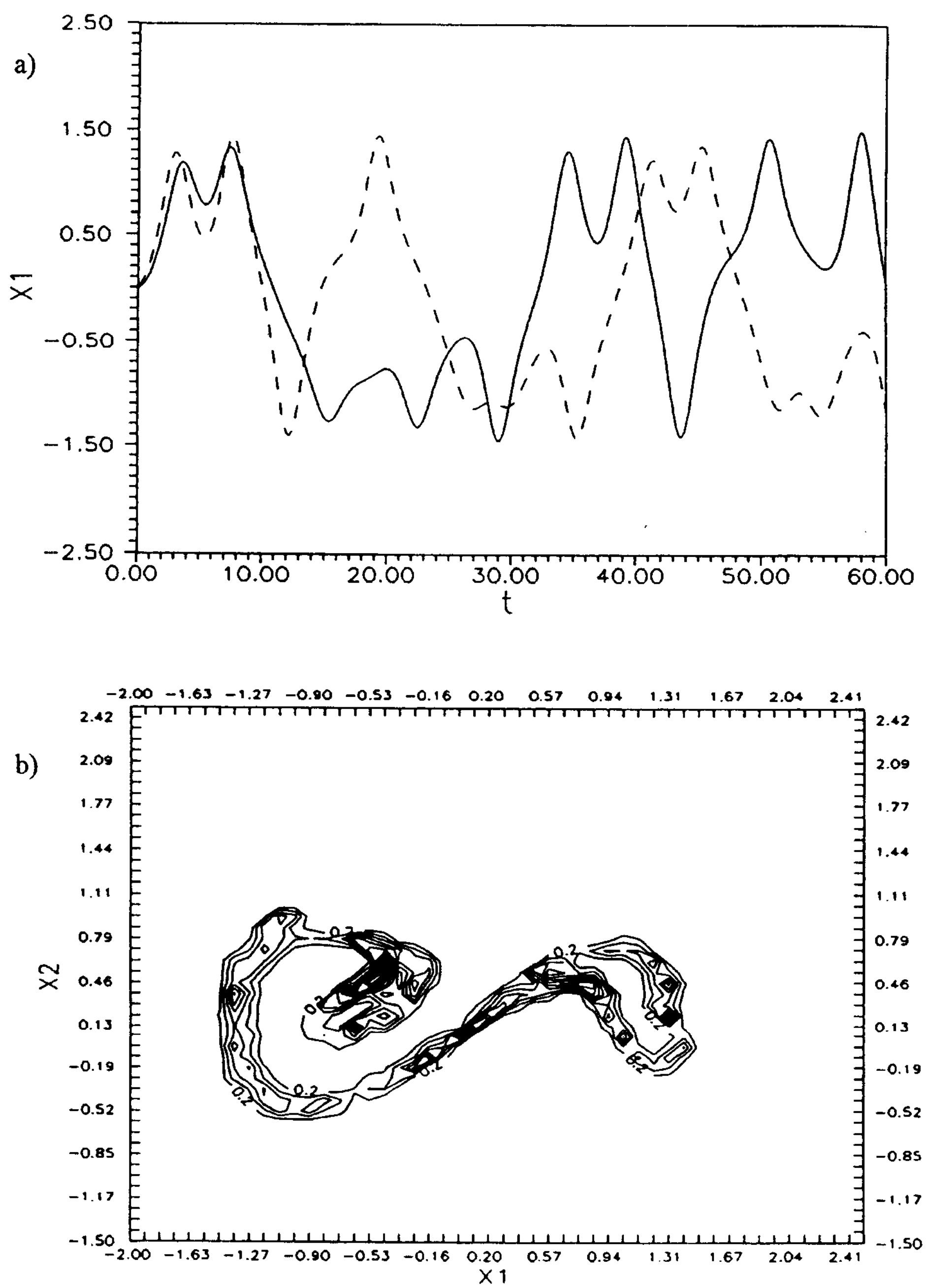

Figure 4. Trajectory vs. density: (a) solid and dashed lines represent chaotic response trajectories with initial conditions at (0, 0) and (0, 0.1), respectively; (b) contour plot of density of corresponding attractor on physical phase plane; $(A, \omega, c, k_1, k_3) = (0.3, 1.0, 0.185, 1.0, 1.0)$.

density is a means for further study of stochastic properties of deterministic chaos as examined in the following sections.

6.1.2. *Evolution of Sensities*

Periodic samples of the evolution of the density in state space governing the responses of the homoclinic and heteroclinic regions are examined based on the numerical experiment described in Section 6.1.1. Figures 5 and 6 show the marginal densities corresponding to typical homoclinic and heteroclinic chaotic responses, respectively, sampled with various (phase) shifts ($\Psi = 0.0 \times T, 0.2 \times T, 0.4 \times T, 0.6 \times T, 0.8 \times T$ and $1.0 \times T$). It is shown that the probability densities are practically identical when the shift is at integer multiples of the forcing period, i.e., $0.0 \times T$ and $1.0 \times T$. The densities appear to be symmetric about the half-cycle time point, i.e., at 0.2×T and $0.8 \times T$, and at $0.4 \times T$ and $0.6 \times T$, the density are practically identical. Thus periodicity in the evolution is demonstrated.

6.1.3. *Asymptotic Stability of Densities*

The Foguel Alternative Theorem is equally applicable to deterministic transformations as indicated in Section 5.3.3 and the asymptotic stability of densities of deterministic chaos is interpreted here accordingly. Because of the existence of periodicity in the evolution of probability density, based on part (b) of the Foguel Alternative Theorem (Equation (10b)), a stationary density (and hence an invariant measure) exists for both deterministic homoclinic and heteroclinic regions. For this example, the time shift in the Foguel Alternative Theorem, t_0, is equal to the excitation period, T. The corresponding time-averaged probability densities (invariant measure) are shown in Figure 7. The numerical results demonstrate and the Foguel Alternative Theorem assures that the deterministic chaotic responses for both regions in the attracting domains are bounded and asymptotically stable.

As numerical results demonstrate that the unpredictability and sensitivity of individual chaotic response trajectory are difficult to take into account in deterministic engineering designs. However, by introducing an initial distribution and through a deterministic transformation defined by the semigroup of Frobenius–Perron operators (Section 2.1), invariant properties of deterministic chaos can be extracted via density on the Poincaré section. Complete information about the existing attractor (if no other co-existing attractors) is provided by the density and the existence of invariant measure can be computed by time-averaging the densities over a forcing period, based on the Foguel Alternative Theorem. In mathematical terms, the deterministic chaotic attractor is of Lebesque measure zero in the phase space (a space of practical interest for conventional reliability analysis). Also, for the case of coexisting response attractors, which is one of the features of nonlinear dynamics, the physical measure may depend on the initial distribution, and only partial information about the physical phase space is provided. An example of the dependence between the initial distribution and coexisting attractors is demonstrated (Figure 8). Figure 8a shows two initial uniform distributions with mean located at $(0, 0)$ and $(1, 2)$, respectively. The corresponding response attractors (chaotic and periodic) are shown in Figures 8b and c, different stochastic properties of each attractor are observed.

The aforementioned drawbacks, i.e. measure zero and partial information of phase space, of directly incorporating deterministic chaos in engineering practice can be ameliorated by including stochastic components in the excitation, which in reality better describes the random nature of environmental loads. Results of engineer interest based on stochastic analysis are demonstrated and discussed in the following section.

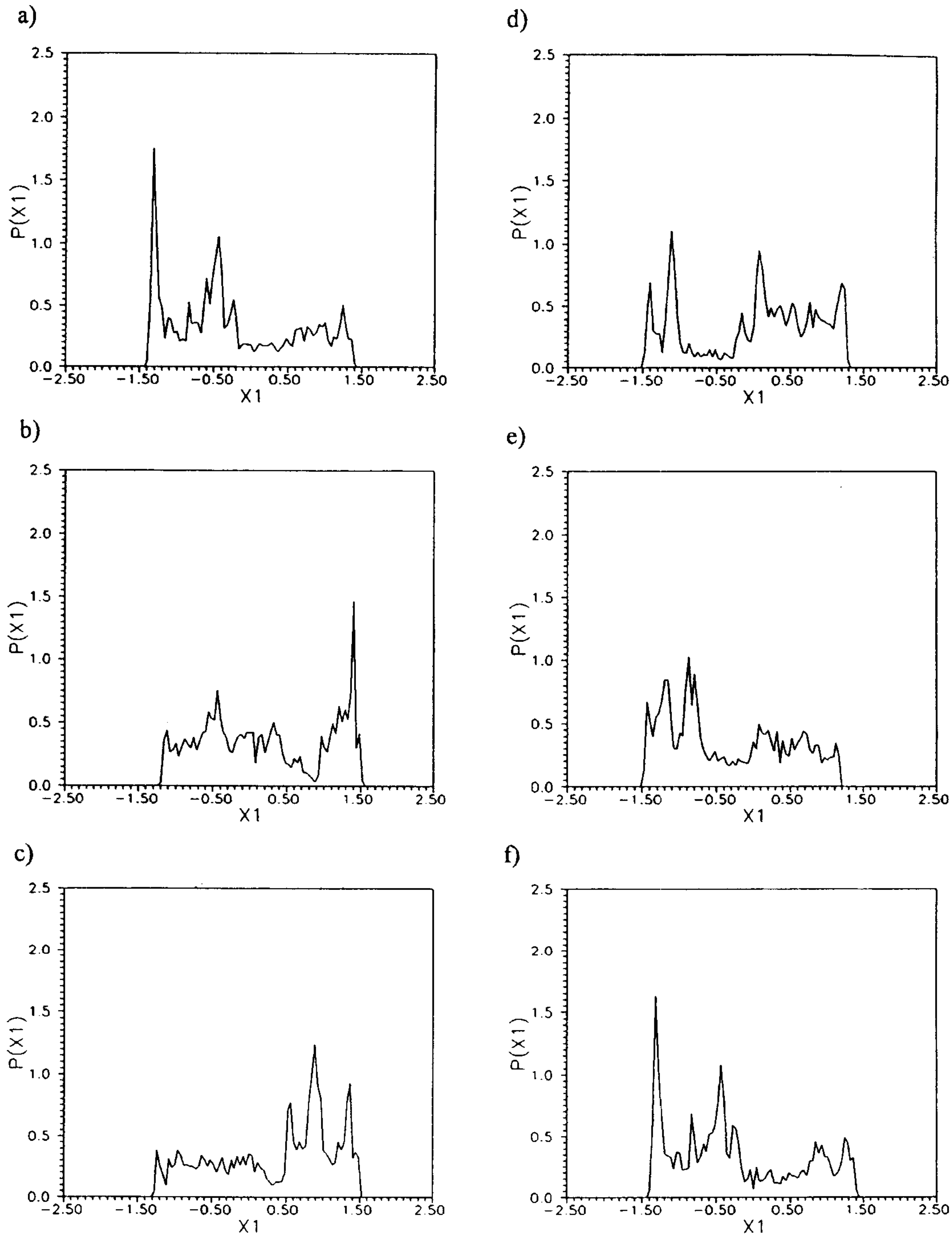

Figure 5. Periodicity in evolution of density of homoclinic chaos: Ψ = (a) $0.0 \times T$, (b) $0.2 \times T$, (c) $0.4 \times T$, (d) $0.6 \times T$, (e) $0.8 \times T$, and (f) $1.0 \times T$; $(k_1, k_3, c, A, \omega) = (1, 1, 0.185, 0.3, 1.0)$.

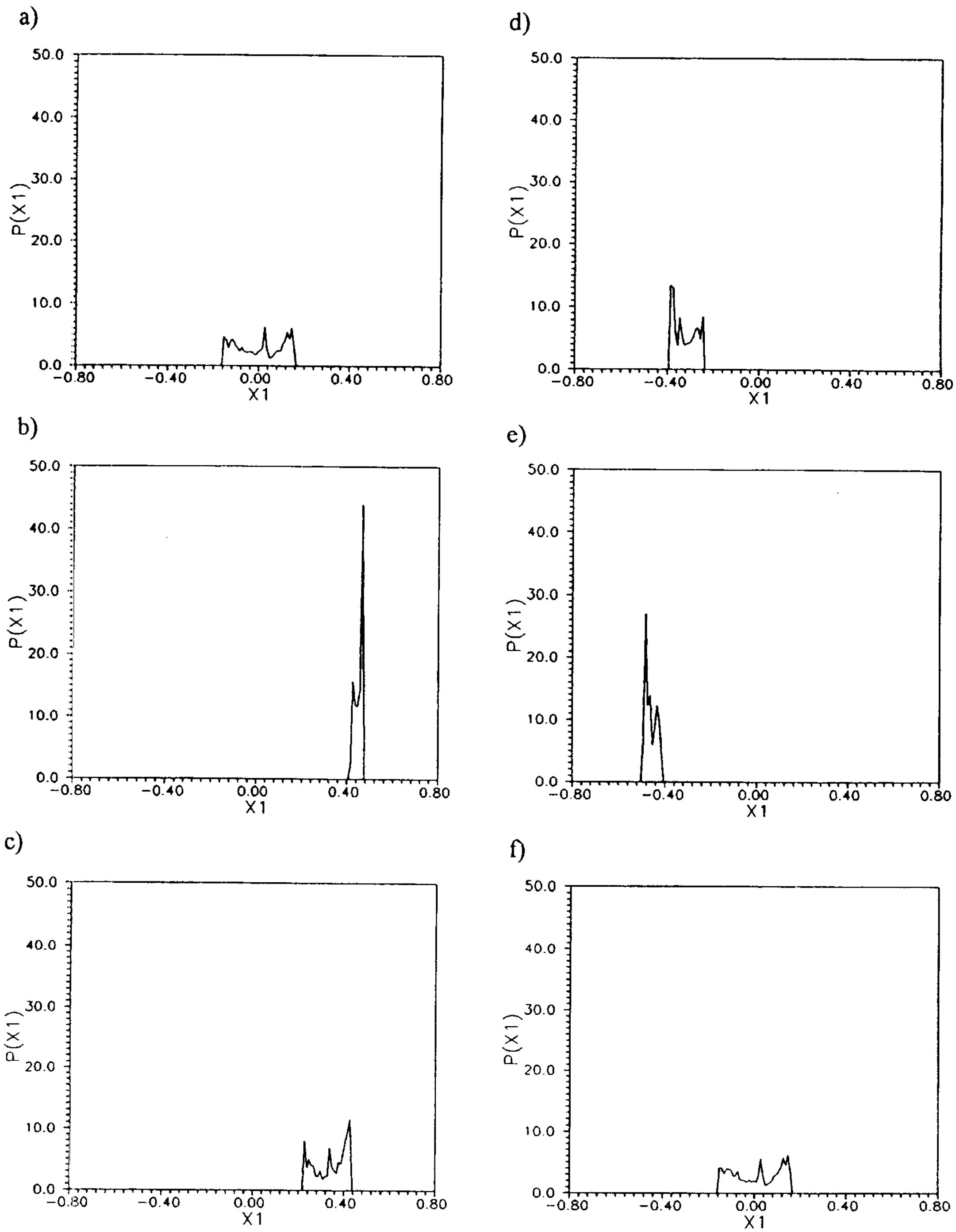

Figure 6. Periodicity in evolution of density of heteroclinic chaos: Ψ = (a) $0.0 \times T$, (b) $0.2 \times T$, (c) $0.4 \times T$, (d) $0.6 \times T$, (e) $0.8 \times T$, and (f) $1.0 \times T$; $(k_1, k_3, c, A, \omega) = (1, 4, 0.4, 0.115, 0.5255)$.

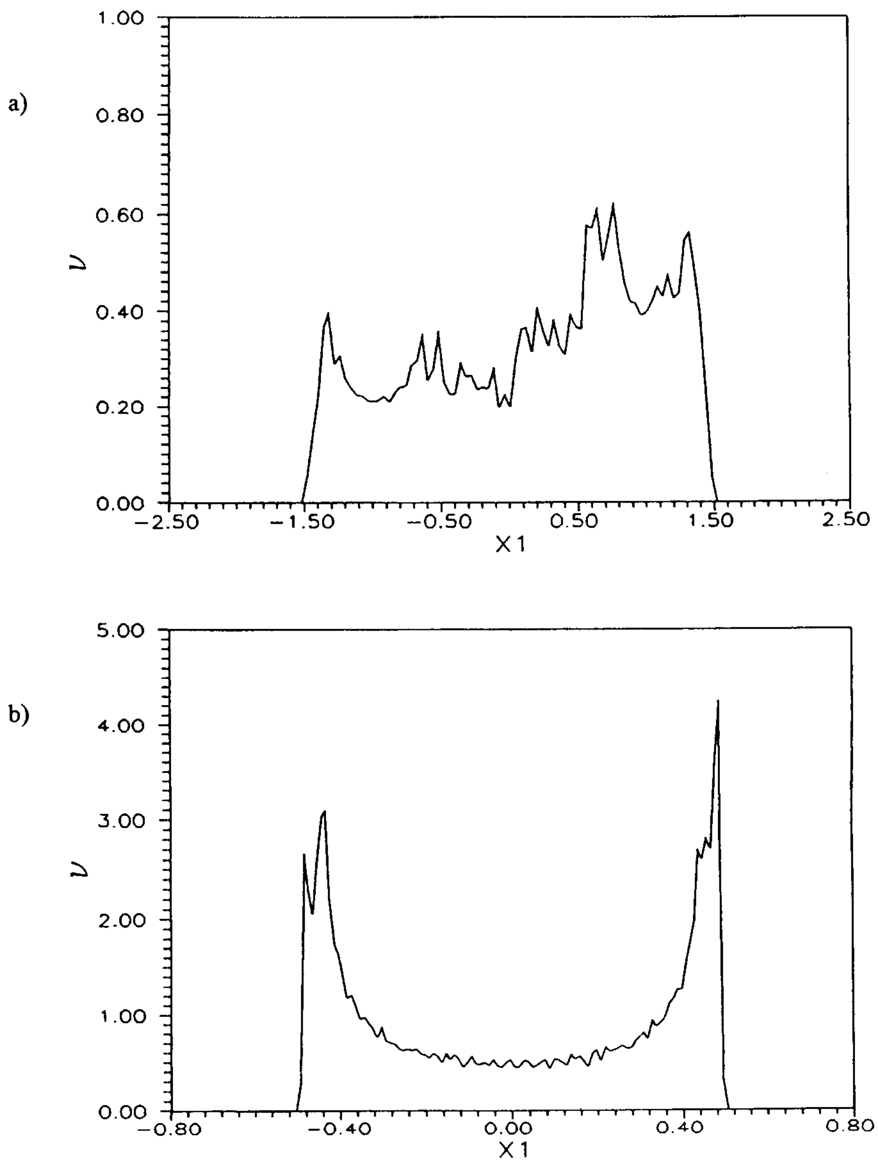

Figure 7. Invariant measure (time-averaged density) of deterministic chaos: (a) homoclinc with $(A, \omega, c, k_1, k_3) = (0.3, 1.0, 0.185, 1.0, 1.0)$; (b) heteroclinic with $(A, \omega, c, k_1, k_3) = (0.115, 0.5255, 0.4, 1.0, 4.0)$.

6.2. Randomly Perturbed Chaotic Responses

The advantages of including random perturbations as a component of the excitation in a highly nonlinear (chaotic) system have been briefly discussed in Section 5.2, and a direct application of stochastic analysis is rendered. Sample randomly perturbed chaotic responses from both homoclinic and heteroclinic dynamics (Equations (2a) and (2b)) are examined via evolution of density to demonstrate their asymptotic behaviors. Because of their distinct asymptotic stabilities, the homoclinic and heteroclinic dynamics are examined individually and discussed in the following sections.

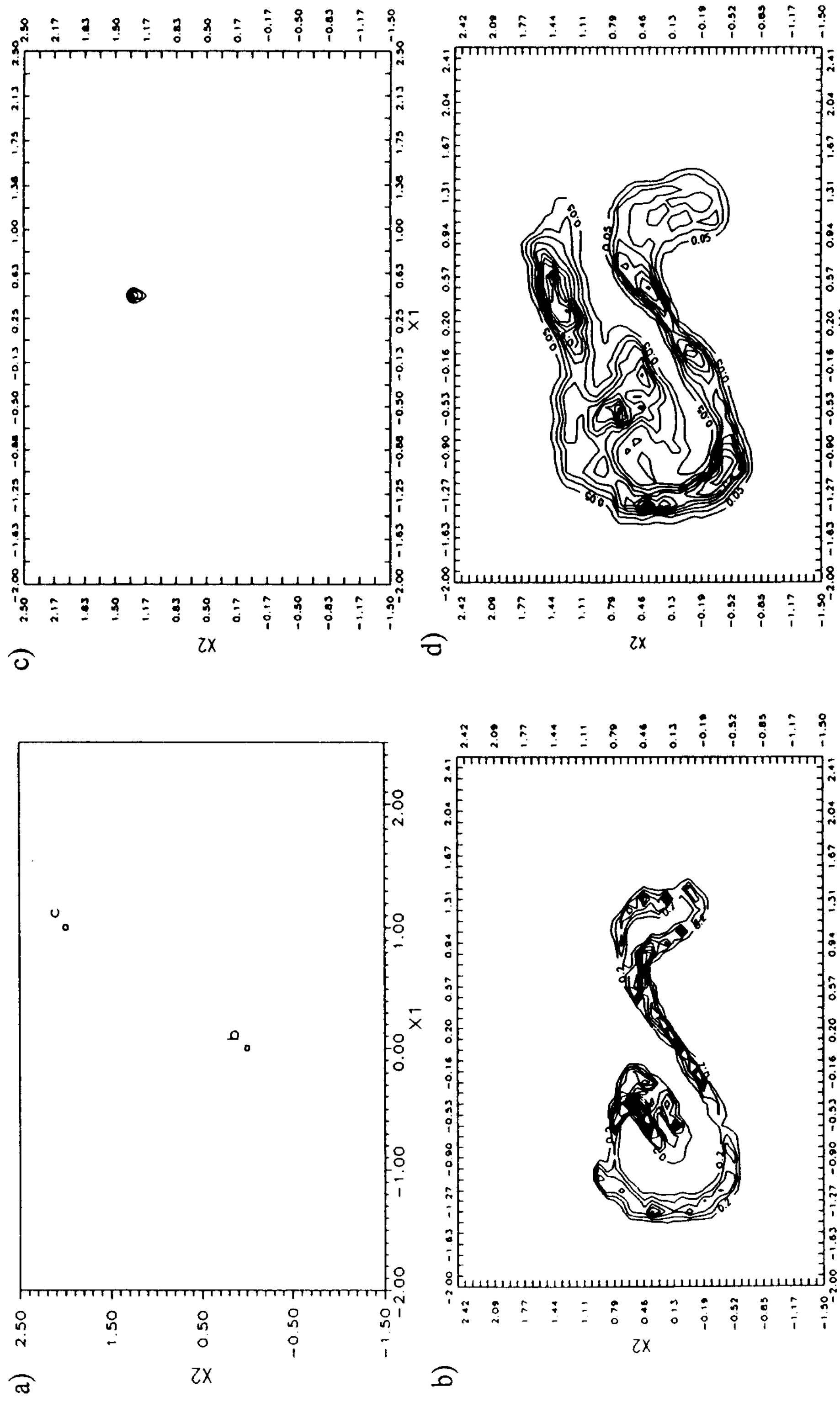

Figure 8. Coexisting response attractors (homoclinic region): (a) initial distributions center at (0, 0) and (1, 2), (b) deterministic chaotic attractor with I.C. (0, 0), (c) deterministic periodic attractor with I.C. (1, 2), and (d) unique density representing coexisting attractors with random excitation component (κ = 0.003); $(A, \omega, c, k_1, k_3) = (0.3, 1.0, 0.185, 1, 1)$.

6.2.1. *Homoclinic Dynamics*

Global information about the physical phase space provided by densities is an important feature of practical engineering interest. The density is obtained by solving the FPE (Equation (9a)) using a path-integral solution procedure, and sampled on the Poincaré section (multiple times of the forcing period). As shown in Figure 8, the density evolves into a steady state which contains both (randomly perturbed) periodic and chaotic motions (Figure 8d). Distribution of probability density in state space indicates the relative strengths of the coexisting attractors [18]. Transitions between the coexisting periodic and chaotic responses may exist in practice. The unique (probability) density, containing both 'periodic' and 'chaotic' motions can be considered as a physical measure in engineering applications (e.g., [28]).

Periodic samples of the evolution of the density in state space governing the responses of the homoclinic region are shown in Figure 9. Starting from the quiescent condition (0, 0) (Figure 9a), the density gradually evolves over the chaotic attracting domain (Figures 9b–d). The density finally settles into a steady state which is invariant on the Poincaré section, thus periodicity in the evolution is assured. Based on part (b) of the Foguel Alternative Theorem, the presence of periodicity in the evolution of the density implies the existence of an invariant density, which can be obtained by averaging the steady-state density over one forcing period (Equation (10b)). The existence of invariant measure (Figure 10) ensures that the system is asymptotically stable.

It is observed that the stochastic analysis presented here via the FPE formulation captures the essential behaviors and the stable attractors (both periodic and chaotic) with relatively little computational efforts (in terms of machine execution time, once the numerical code is developed). On the other hand, in a deterministic analysis, the amount of analytical and computational efforts involved in identifying the co-existing attractors (via large order of trial simulations) and examining their stability is often substantially higher in comparison (in terms of engineering hours).

6.2.2. *Heteroclinic Dynamics*

Evolution of the first few cycles of the probability density of randomly perturbed chaotic responses of the heteroclinic region is obtained by numerically solving the FPE (Equation (9b)) and presented on the Poincaré section in Figure 11. Starting with the quiescent initial condition (0, 0) (Figure 11a), the probability density spreads widely over the phase space after a cycle of the forcing period (Figure 11b), and the probability mass keeps diverging toward $\pm\infty$ within the first few cycles (Figures 11c–11d). Continuing integration of the evolution of the probability density (not presented in the figure) shows that spreading of the density continues with increasing number of excitation cycles and eventually all probability mass is swept out of the region of interest. (Note that the stable region of interest is a function of the excitation details and system parameters such as damping coefficient). These numerical results are consistent with the sweeping property (i.e., part (a)) of the Foguel Alternative Theorem (Section 5.3). Because a stationary density (invariant measure) does not exist within the region of interest, all probability mass will eventually escape (be swept) from the region. Sweeping of the probability mass from the stable, bounded-motion region indicates that the heteroclinic dynamics is asymptotically unstable under the presence of random perturbations, thus all perturbed response trajectories near the separatrix will eventually diverge to unbounded motions.

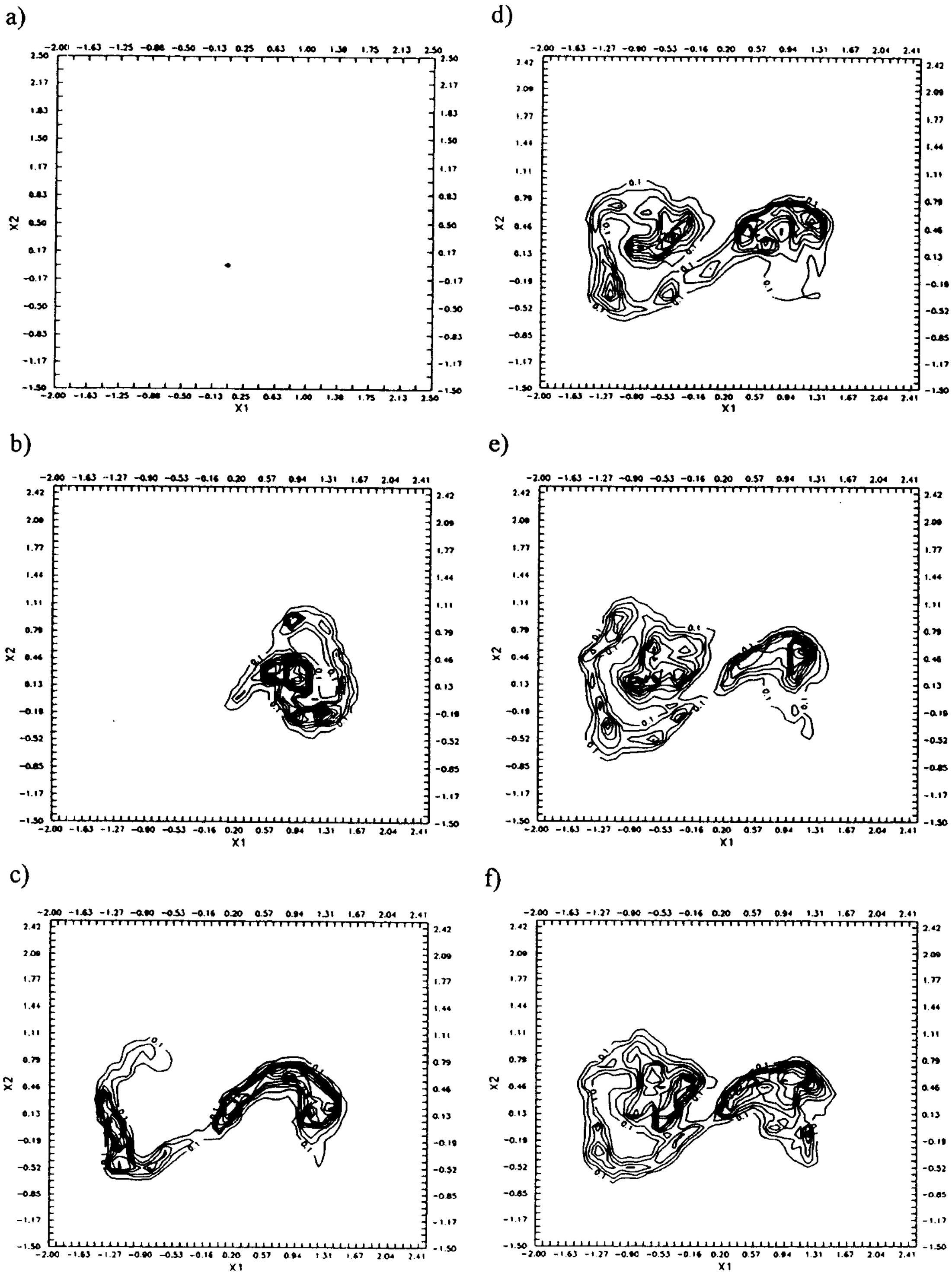

Figure 9. Evolution of density of randomly perturbed chaos (homoclinic region): contour maps at (a) initial conditions (0.0, 0.0), (b) 1st, (c) 2nd, (d) 4th, (e) 18th, and (f) 20th cycle of forcing period, $(A, \omega, c, k_1, k_3, \kappa) = (0.27, 1.0, 0.185, 1, 1, 0.003)$.

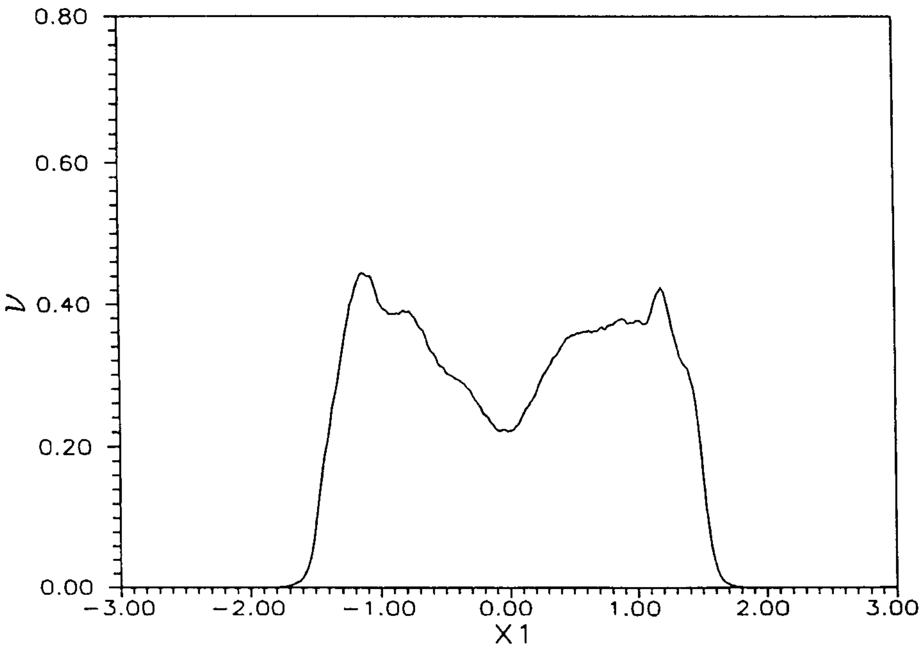

Figure 10. Invariant measure (time-averaged density) of randomly perturbed chaos; homoclinic dynamics with $(A, \omega, c, k_1, k_3, \kappa) = (0.3, 1.0, 0.185, 1.0, 1.0, 0.003)$.

7. Engineering Application

It has been shown above that the probability measure is the unique normalized measure for nonlinear stochastic systems and the associated probability densities provide a means for practical reliability analyses. The asymptotic behavior of the density is characterized by the Foguel Alternative Theorem, i.e., the density is either invariant or sweeping in time. For the case that an invariant measure exists (the time-average probability density, cf. Equation (10b)), various models have been developed to approximate the frequency of response trajectories up-crossing at a given amplitude. With further statistical assumptions, the probability of large excursions in system responses can be estimated and serve as a reliability index for system performance (e.g., [30]). On the other hand, for the case with a sweeping probability density, the probability of system response trajectories escaping from the 'defined' domain will eventually become unity. For this type of response, the transient behavior of the evolution of the probability density become important in evaluating the system performance. Despite the two distinct asymptotic density behaviors (i.e., invariant and sweeping), the reliability of a general class of nonlinear stochastic systems with either asymptotic density behavior can be reflected by a unified index, the first passage time, as demonstrated in the following.

7.1. Reliability by First Passage Time

Reliability of a system is defined as the system behaves in a designated 'safe' domain within a designed life span. The reliability can be translated into a probabilistic description [22]

$$P_S(\tau) = P\{X(t) \in D, \quad o \leq t \leq \tau\}, \tag{11a}$$

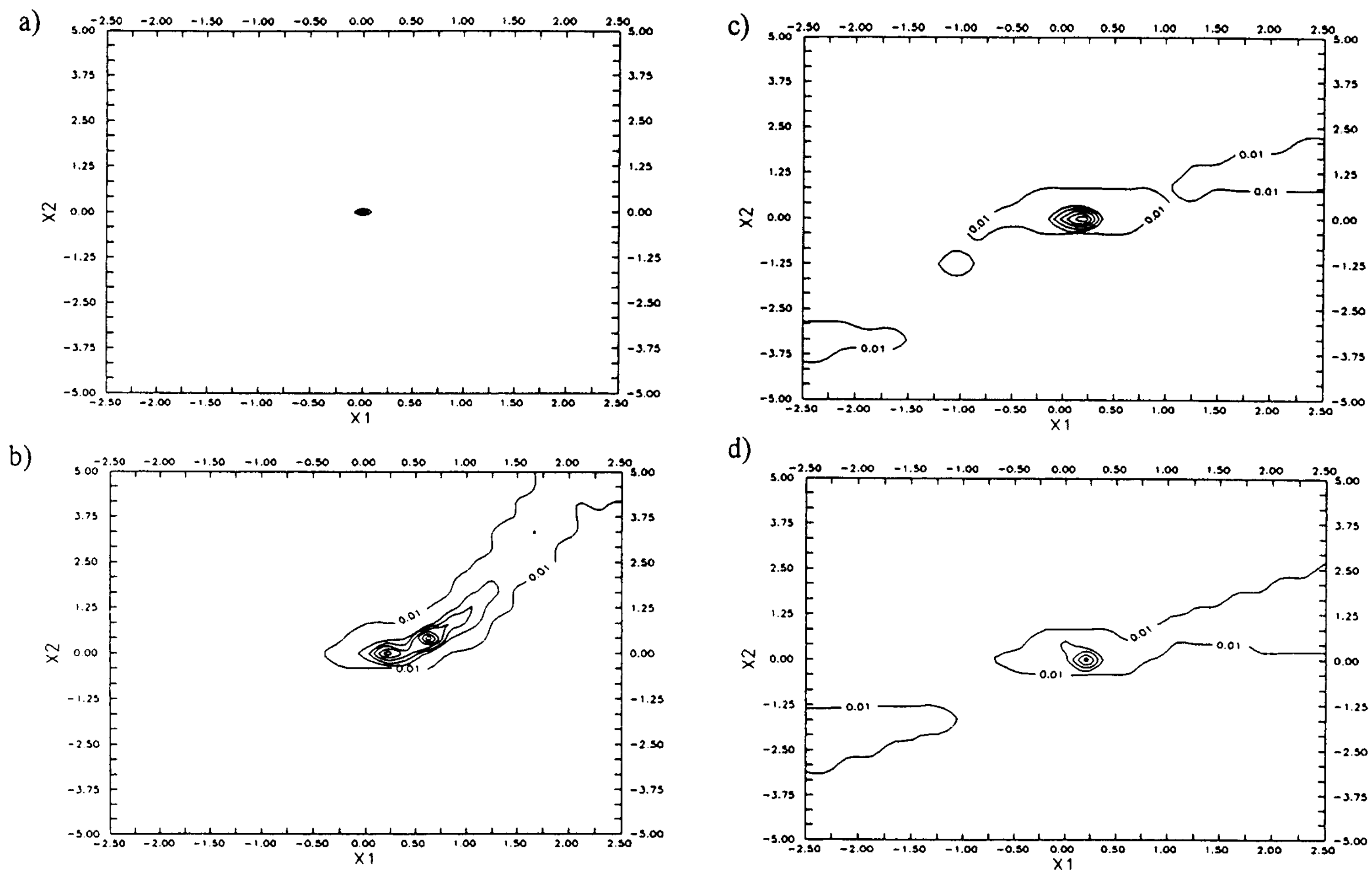

Figure 11. Sweeping property of density (randomly perturbed heteroclinic dynamics) at: (a) initial conditions (0.0, 0.0), (b) 1st, (c) 4th, and (d) 6th cycle of forcing period, $(A, \omega, c, k_1, k_3, \kappa) = (0.115, 0.5255, 0.4, 1, 4, 0.005)$.