*Proceedings of the Third (1993) International Offshore and Polar Engineering Conference*
*Singapore, 6-11 June 1993*

*ISBN 1-880653-05-2 (Set); ISBN 1-880653-08-7 (Vol III)*

# Nonlinear Motions of Tethered Floating Buoys

*John W. Leonard*
University of Connecticut
Storrs, Connecticut, USA

*Krisnaldi Idris and Solomon C.S. Yim*
Oregon State University
Corvallis, Oregon, USA

## ABSTRACT

The equations of motion for small tethered buoys floating in a nonlinear wave field have been developed. The coupling between rotational and translational degrees of freedom is included in the equations and a three-dimensional response is assumed. The floating buoy is treated as one boundary condition of the governing differential equations for the mooring line coupled buoy-mooring problem. Hydrodynamic forces are calculated from the relative-motion form of the Morrison equation.

## INTRODUCTION

In this paper the coupling effects of rotational degrees of freedom of tethered floating buoys with the governing equations of the tether are considered. The cable algorithm is described in the following section. The equations of motion for tethered floating buoys in terms of the six degrees of freedom in translation and rotation, which constitute the boundary conditions for one end of the tether, are developed. An algorithm for quasi-linearization of those boundary conditions, which are used in determining the tether motions and buoy rotations for the coupled nonlinear system, is developed and presented in a subsequent section. Validation of the methodology is provided in the final section.

Buoys and their moorings are considered in this work to be classified as small bodies for which the relative-motion Morison equation may be adopted (Sarpkaya and Isaacson, 1981). A coupled analysis is needed for this ocean structure, since the motion of the buoy affects the motion of the mooring and *visa versa* (Berteaux, 1976).

## CABLE ALGORITHM

An iterative algorithm of dynamic analysis of hydro-dynamically loaded cable has been developed by Chiou and Leonard (1991) in which the problem is formulated as a two point boundary value problem. The boundary value problem is then transformed into an iterative set of quasi-linearized boundary value problems, which is then decomposed (Atkinson, 1989) into a set of initial value problems so that spatial integration may be performed along the cable (Sun et al., 1993). Solutions of each initial value problem are recombined so as to always satisfy boundary conditions; then solutions of the boundary value problem are obtained by successive iteration. In decomposing the boundary value problem into a set of initial value problems, one expresses the solution as a linear combination of homogenous solutions (${}^j\dot{X}_{Ti}$ and ${}^jT_i$) and particular solutions (${}^0\dot{X}_{Ti}$ and ${}^0T_i$).

$$\dot{X}_{T_i} = {}^\circ\dot{X}_{T_i} + \alpha_j\ {}^j\dot{X}_{T_i} \tag{1}$$

$$T_i = {}^\circ T_i + \alpha_j\ {}^jT_i \tag{2}$$

where $\alpha_j$'s are undetermined coefficients, $\dot{X}_{Ti}$'s are components of cable velocity and $T_i$'s are components of cable tension.

Several kinds of boundary condition may be applied on both ends of the cable. At one end the mooring cable may be held fixed to the ocean floor and thereby requires zero velocity at the boundary at all times

$$\dot{X}_{Ti}(t) = 0 \tag{3}$$

At the other end a floating buoy is attached and buoy/body boundary conditions are applied. In this case the equations of motion for tethered floating buoys serve as a boundary condition.

## EQUATIONS OF MOTION FOR BUOY

A definition sketch of a buoy floating on the moving water surface and connected by a tether to the ocean bottom is depicted

[1]ISOPE Member

in Figure 1. Two coordinate systems, a moving system attached to the buoy and a fixed (world) system, are used. As the buoy moves from an initial static equilibrium position to a position at time t, the position vector of a general point P in the buoy in terms of world coordinates $x_i$ is

$$\underline{X}_P = \underline{X}_G + \underline{Z}_{GP} + \underline{R} \times \underline{Z}_{GP} \qquad (4)$$

and

$$\underline{X}_G = \underline{S} + \underline{D} \qquad (5)$$

Then, since the acceleration and velocity at G are $\ddot{\underline{U}}$ and $\dot{\underline{U}}$ and the rotational acceleration and velocity about point G are $\ddot{\underline{R}}$ and $\dot{\underline{R}}$ respectively, the buoy acceleration and velocity at a general point P are

$$\ddot{\underline{X}}_P = \ddot{\underline{U}} + \ddot{\underline{R}} \times \underline{Z}_{GP} \qquad (6)$$

$$\dot{\underline{X}}_P = \dot{\underline{U}} + \dot{\underline{R}} \times \underline{Z}_{GP} \qquad (7)$$

The vector sum of the forces acting on the buoy and their moments about point G must equal the inertial force/moment vector for the buoy, $[M]\,\ddot{\underline{D}}$, where $[M]$ is the diagonal mass matrix for the buoy, and $\underline{D}$ is the displacement of the center of gravity. Thus

$$[M]\ddot{\underline{D}} = \underline{F}_W - \underline{T} + \underline{W}_{DRY} + \underline{B} + \underline{F}_I + \underline{F}_D + \underline{F}_K \qquad (8)$$

The seven force components on the right hand side of Eq. (8) are, wind or other force $\underline{F}_W$, tether tension T, dry weight $W_{DRY} = M\, g\, \underline{e}_1$, buoyancy $\underline{B}$, hydrodynamics inertia force $\underline{F}_I$ and hydrodynamic drag $\underline{F}_D$, respectively. Components of individual matrices and vectors are written in the Appendix.

The equation of motion may be separated into translational and rotational components. The translational components of the equation of motion serve as three boundary conditions for the tether point tensions and translational velocities (at point T) and the rotational components serve as three auxiliary differential equations for the buoy rotations $R_i$. To implement these boundary conditions one must first express $\dot{U}_i$ and $\ddot{U}_i$ in terms of velocity and acceleration at the tether point T using Eqs. (6) and (7),

$$\ddot{\underline{U}} = \ddot{\underline{X}}_T - \ddot{\underline{R}} \times \underline{Z}_{GT} \qquad (9)$$

$$\dot{\underline{U}} = \dot{\underline{X}}_T - \dot{\underline{R}} \times \underline{Z}_{GT} \qquad (10)$$

Now the buoy equations of motion may be written, in indicial notation, as

$$\begin{aligned} &-(M_{(i)} + \rho V_{wet} C_{A(i)(i)})(\ddot{X}_{Ti} - \epsilon_{ijk} Z_{GTk} \ddot{R}_j] \\ &- \rho V_{wet} C_{A(i)(j+3)} \ddot{R}_j + F_{wi} - T_i + (M_{(1)} \\ &- \rho V_{wet}) g \delta_{1j} + (1 + C_{A(i)(i)}) \rho V_{wet} \dot{V}_i \\ &+ N_{D(i)}(V_i - \dot{X}_{Ti} - \epsilon_{ijk} Z_{TBk} \dot{R}_j) = 0 \end{aligned} \qquad (11)$$

$$\begin{aligned} &-[I_{ij} + C_{A(i+3)(j+3)} \rho V_{wet}] \ddot{R}_j \\ &- \rho V_{wet} C_{A(i+3)(j)} [\ddot{X}_{Ti} - \epsilon_{ijk} Z_{GTk} \ddot{R}_j] \\ &- [N_{D(j)} W_{ij}] \dot{X}_{Tj} - [\epsilon_{kjn} N_{D(k)} Z_{GBn} W_{ik}] \dot{R}_j \\ &+ [Z_{GWi} F_{Wj} - Z_{GTi} T_j - Z_{GBi} B_j \delta_{1j} \\ &- \delta_{ij} (Z_{GWk} F_{Wk} - Z_{GTk} T_k - Z_{GBk} B_k \delta_{1k})] R_j \\ &+ \epsilon_{ijk} [Z_{GWj} F_{Wk} - Z_{GTj} T_k - M_{Fjk}] \\ &- \rho g I_{wpj} (R_j - \zeta_j) = 0 \end{aligned} \qquad (12)$$

where the distance from T to B is

$$Z_{TBi} = Z_{GBi} - Z_{GTi} \qquad (13)$$

$$W_{ik} = \epsilon_{ink} Z_{GBn} + Z_{GBi} R_k - Z_{GBk} R_i \qquad (14)$$

$$\begin{aligned} M_{Fik} &= (Z_{GBi} + C_{A(i+3)(k)}) \rho V_{wet} \dot{V}_k \\ &+ (N_{D(k)} V_k - \rho g V_{wet} \delta_{1k}) Z_{GBi} \end{aligned} \qquad (15)$$

and the magnitude of relative velocity at point B is

$$\begin{aligned} Q = [&(V_i - \dot{X}_{Ti} - \epsilon_{ijk} Z_{TBk} \dot{R}_j) \\ &(V_i - \dot{X}_{Ti} - \epsilon_{imn} Z_{TBn} \dot{R}_m)]^{1/2} \end{aligned} \qquad (16)$$

Equations (11) and (12) are second-order ordinary differential equations in time. Given solutions $X_{Ti}^*$, $\dot{X}_{Ti}^*$, $\ddot{X}_{Ti}^*$, $R_i^*$, $\dot{R}_i^*$, $\ddot{R}_i^*$ at time $t^*$, accelerations at time $t = t^* + \Delta t$ are approximated as (Sun et al., 1993)

$$\ddot{x}_{Ti} = (\dot{X}_{Ti} - \dot{X}_{Ti}^*)/(\alpha \Delta t) - \gamma \ddot{X}_{Ti}^* \qquad (17)$$

$$\ddot{R}_i = (\dot{R}_i - \dot{R}_i^*)/(\alpha \Delta t) - \gamma \ddot{R}_i^* \qquad (18)$$

where $\alpha = 1/2$ for implicit integration and $\gamma = (1-\alpha)/\alpha$. Using the same formulation, the translational and rotational displacements can be expressed as

$$X_{Ti} = \alpha \Delta t (\dot{X}_{Ti} + \gamma \dot{X}_{Ti}^*) + X_{Ti}^* \qquad (19)$$

$$R_i = \alpha \Delta t (\dot{R}_i + \gamma \dot{R}_i^*) + R_i^* \qquad (20)$$

Then, upon substitution of Eqs. (17) through (20) into Eqs. (11) and (12), quasi-static nonlinear equations at time t are obtained as

$$f_i(\dot{X}_{Tj}, \dot{R}_j, T_j) = 0 \qquad (21)$$

for force equilibrium, and

$$h_i(\dot{X}_{Tj}, \dot{R}_j, T_j) = 0 \qquad (22)$$

for moment equilibrium.

## QUASI-LINEARIZATION OF BUOY BOUNDARY CONDITIONS

Equations (21) and (22) are the nonlinear boundary conditions for the tether attached to point T. Following the iterative scheme for solving the cable equations for the tether as described by Chiou and Leonard (1991) and Sun et al. (1993), the nonlinear boundary conditions need to be quasi-linearized. If rotational velocities are included, as in Eqs. (21) and (22), the Newton-Raphson method (Atkinson, 1989) can be used to determine improved estimates $T_i$, $\dot{X}_{Ti}$ and $\dot{R}_i$, given prior estimates $T_i'$, $\dot{X}_{Ti}'$ and $\dot{R}_i'$. Taking Taylor series expansions of Eqs. (21) and (22) about the functions $f_i'$ and $h_i'$ evaluated at $T_i'$, $\dot{X}_{Ti}'$ and $\dot{R}_i'$ with respect to increments $(T_i - T_i')$, $(\dot{X}_{Ti} - \dot{X}_{Ti}')$, and $(\dot{R}_i - \dot{R}_i')$, one writes

$$f_i = 0 = f_i' + J_{FXij}'(\dot{X}_{Tj} - \dot{X}_{Tj}') + J_{FTij}'(T_j - T_j') + J_{FRij}'(\dot{R}_j - \dot{R}_j') \tag{23}$$

$$h_i = 0 = h_i' + J_{HXij}'(\dot{X}_{Tj} - \dot{X}_{Tj}') + J_{HTij}'(T_j - T_j') + J_{HRij}'(\dot{R}_j - \dot{R}_j') \tag{24}$$

Components of the Jacobian matrices are written in the Appendix.

Now, Eqs. (23) and (24) are six linear algebraic equations for the nine unknowns $T_i$, $\dot{X}_{Ti}$ and $\dot{R}_i$ in terms of prior estimates $T_i'$, $\dot{X}_{Ti}'$ and $\dot{R}_i'$. The six unknowns $T_i$, $\dot{X}_{Ti}$ are related to the three boundary conditions at the other end of the tether through the cable differential equations. Then, upon substitution of (1) and (2), one obtains six equations to be solved for six unknowns $\alpha_i$ and $\dot{R}_i$.

Rewrite Eqs. (23) and (24) in matrix form as

$$\begin{bmatrix} K_{FF}' & K_{FR}' \\ K_{RF}' & K_{RR}' \end{bmatrix} \begin{Bmatrix} \alpha \\ \dot{R} \end{Bmatrix} = \begin{Bmatrix} P_F' \\ P_R' \end{Bmatrix} \tag{25}$$

where the submatrices are given by

$$[K_{FF}'] = [J_{FXik}'][\{^1\dot{X}_{Tk}\},\{^2\dot{X}_{Tk}\},\{^3\dot{X}_{Tk}\}] + [J_{FTik}'][\{^1T_j\},\{^2T_j\},\{^3T_j\}] \tag{26}$$

$$[K_{RF}'] = [J_{HXik}'][\{^1\dot{X}_{Tk}\},\{^2\dot{X}_{Tk}\},\{^3\dot{X}_{Tk}\}] + [J_{HTik}'][\{^1T_j\},\{^2T_j\},\{^3T_j\}] \tag{27}$$

$$[K_{FR}'] = [J_{FRik}'] \tag{28}$$

$$[K_{RR}'] = [J_{HRij}'] \tag{29}$$

$$\{P_F'\} = -\{f_i'\} + [J_{FXij}'](\{\dot{X}_{Tj}'\} - \{^o\dot{X}_{Tj}\}) + [J_{FTij}'](\{T_j'\} - \{^oT_j\}) + [J_{FRij}']\{\dot{R}_j'\} \tag{30}$$

$$\{P_R'\} = -\{h_i'\} + [J_{HXij}'](\{\dot{X}_{Tj}'\} - \{^o\dot{X}_{Tj}\}) + [J_{HTij}'](\{T_j'\} - \{^oT_j\}) + [J_{HRij}']\{\dot{R}_j'\} \tag{31}$$

In each iteration particular and homogenous cable equations are integrated from the bottom to the tether point at the buoy, and Eq. (25) can be solved for the parameters $\alpha_i$ and the new estimates to $\dot{R}_i$. Then, the partial solutions are combined to obtain the initial values for a final integration.

## NUMERICAL EXAMPLES

To demonstrate the capability of the present method, two numerical examples were computed. Both examples used 3/8 in. diameter mooring cables with modulus of elasticity $18\times10^6$ psi, dry weight 0.218 lb./ft., normal drag coefficient 1.2, tangential drag coefficient 0.02 and added mass coefficient 1.0.

The first example is an oblate spheroidal buoy with 5 ft horizontal radius, 1.5 ft. vertical radius, and 3378 lb. dry weight. The buoy is tethered with a 360 ft. (unstretched length) mooring in 530 ft. deep water. The loading is a monochromatic wave train 5 ft. high with a 10 sec. period. A definition sketch of this problem is shown in Figure 2a. The calculated pitch and displacement are shown in Figure 3. The corresponding velocities and acceleration are shown in Figures 4 and 5.

The second example is a prolate spheroidal buoy with 1 ft horizontal radius, 12.5 ft. vertical radius, and 3378 lb. dry weight. The buoy is tethered with a 252 ft. (unstretched length) mooring in 300 ft. deep water. The same wave train as in the first example is applied. A definition sketch of this problem is shown in Figure 2b. The calculated pitch and heave and surge displacements, and their corresponding velocities and acceleration are shown in Figures 6, 7 and 8, respectively.

## DISCUSSION AND CONCLUDING REMARKS

A method to compute buoy motion coupled with mooring motion and tension has been developed. The example for oblate spheroid shows the buoy's pitch motion is very close to the wave slope, as expected for a wave follower disc buoy. The prolate spheroidal buoy shows smaller pitch angle compare to the oblate one. Heave displacement for the oblate spheroidal buoy is about the same magnitude as the wave height while the prolate buoy shows smaller magnitude. In both examples the surge motions show a drift in the wave direction.

## REFERENCES

Atkinson, K.E. (1989). An Introduction to Numerical Analysis, 2nd Ed., John Wiley & Sons, New York.

Berteaux, H.O. (1976). Buoy Engineering, John Wiley & Sons, New York.

Chiou, R.B., and Leonard, J.W. (1991). "Nonlinear Hydrodynamic Response of Curved, Singly-Connected Cables," Proceedings, Second International Conference on Computer Modelling in Ocean Engineering, Barcelona, Spain, Sept.30 - Oct.4.

Sarpkaya, T. and Isaacson, M. (1981). Mechanics of Wave Forces on Offshore Structures, Van Nostrand Reinhold Company, New York.

Sun, Y., Leonard, J.W., and Chiou, R.B. (1993). "Simulation of Unsteady Oceanic Cable Deployment by Direct Integration with Suppression," Ocean Engineering, in press.

## ACKNOWLEDGEMENT

This work was sponsored by the U.S. Naval Civil Engineering Laboratory under Contract N47408-90-C-1146 to the Department of Civil Engineering at Oregon State University, Corvallis, OR.

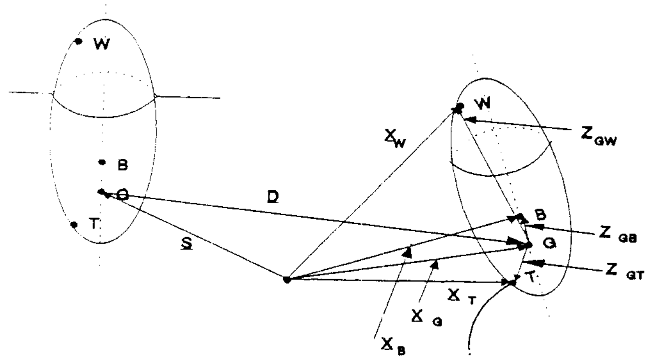


Figure 1. Definition Sketch of Buoy Vectors

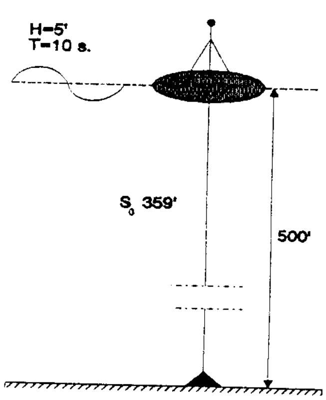


Figure 2a. Oblate Spheroidal buoy

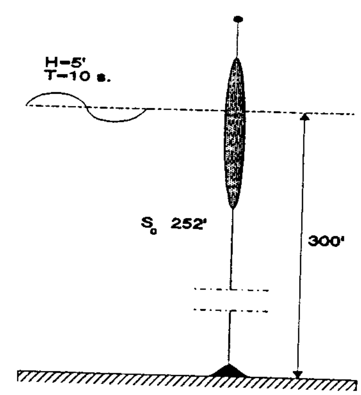


Figure 2b. Prolate Spheroidal Buoy

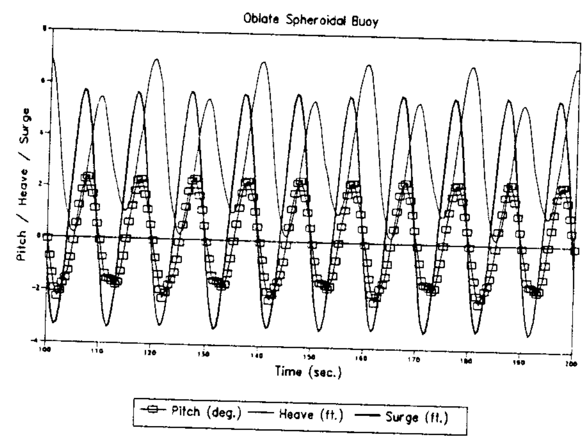


Figure 3. Pitch and Displacements of Oblate Spheroidal Buoy

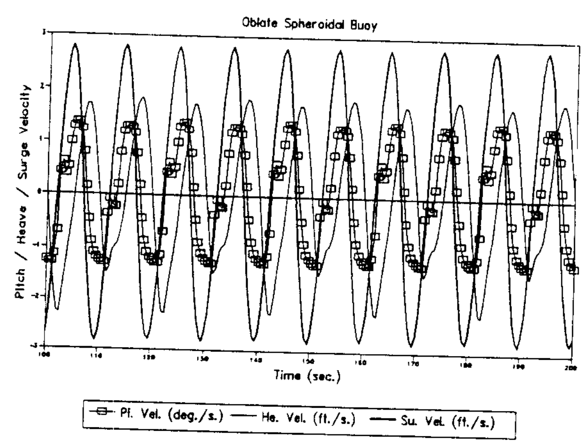


Figure 4. Velocities of Oblate Spheroidal Buoy

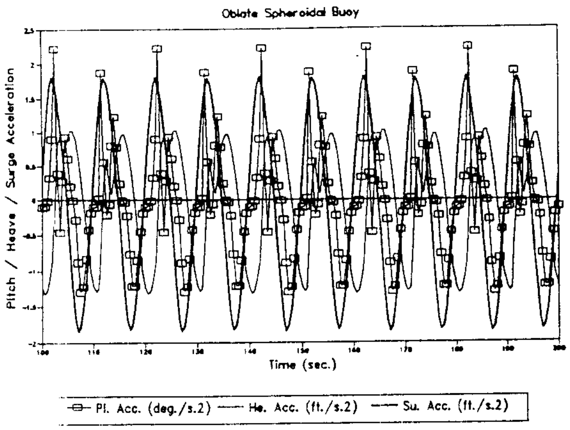


Figure 5. Acceleration of Oblate Spheroidal Buoy

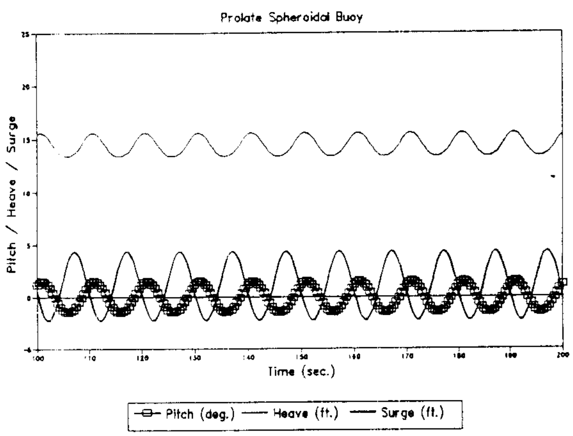


Figure 6. Pitch and Displacement of Prolate Buoy

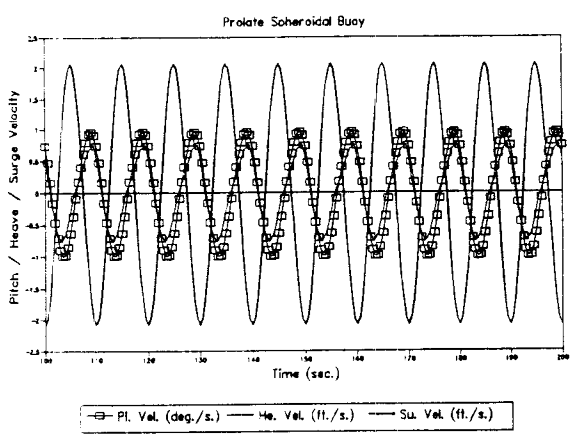


Figure 7. Velocities of Prolate Spheroidal Buoy

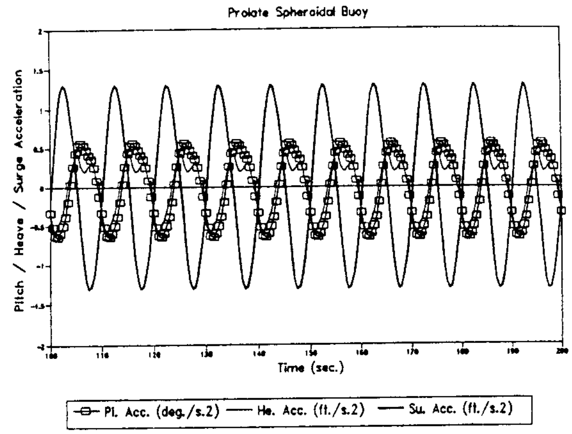


Figure 8. Accelerations of Prolate Spheroidal Buoy

**APPENDIX**

$$[M]\underline{\ddot{D}} = \begin{bmatrix} M & 0 & 0 & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{33} \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{R}_1 \\ \ddot{R}_2 \\ \ddot{R}_3 \end{Bmatrix} \qquad \text{(A1)}$$

where M is mass of the buoy and $I_{(i)(i)}$ is moment of inertia.

$$\underline{F}_I = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & 0 & A_{26} \\ 0 & 0 & A_{33} & 0 & A_{35} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{53} & 0 & A_{55} & 0 \\ 0 & A_{62} & 0 & 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1-\ddot{U}_1 \\ \ddot{v}_2-\ddot{U}_2 \\ \ddot{w}_3-\ddot{U}_3 \\ \ddot{R}_1 \\ \ddot{R}_2 \\ \ddot{R}_3 \end{Bmatrix} \qquad \text{(A2)}$$

where $A_{(i)(j)} = \rho\, V_{WET}\, C_{A(i)(j)}$, in which $\rho$ is water density; $V_{WET}$ is wetted volume; and $C_{A(i)(j)}$ is the added mass coefficient in the $\underline{e}_i$ direction due to motion in the $\underline{e}_j$ direction.

$$\underline{F}_D = \begin{Bmatrix} N_{D1}(u-\dot{U}_1) \\ N_{D2}(v-\dot{U}_2) \\ N_{D3}(w-\dot{U}_3) \\ 0 \\ Z_{GB1}N_{D3}(w-\dot{U}_3) \\ Z_{GB1}N_{D2}(v-\dot{U}_2) \end{Bmatrix} \tag{A3}$$

where $N_{D(i)} = \rho\, A_{D(i)}C_{D(i)}Q/2$, in which $A_{D(i)}$ is the drag area; $C_{D(i)}$ the drag coefficient in the $\underline{e}_i$ direction; and Q the magnitude of $(\underline{V} - \dot{\underline{X}}_B)$.

$$\underline{F}_K = \begin{Bmatrix} \rho V_{WET}\dot{u} \\ \rho V_{WET}\dot{v} \\ \rho V_{WET}\dot{w} \\ 0 \\ Z_{GB1}\rho V_{WET}\dot{w} \\ Z_{GB1}\rho V_{WET}\dot{v} \end{Bmatrix} \tag{A4}$$

$$\underline{B} = \begin{Bmatrix} \rho g V_{WET} \\ 0 \\ 0 \\ 0 \\ \rho g V_{WET} Z_{GB1} R_2 - \rho g I_{WP}(R_2 - \zeta_2) \\ \rho g V_{WET} Z_{GB1} R_3 - \rho g I_{WP}(R_3 - \zeta_3) \end{Bmatrix} \tag{A5}$$

$$\underline{T} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ (Z_{GT2} - R_1 Z_{GT3} + R_3 Z_{GT1})T_3 \\ \quad + (-Z_{GT3} - R_1 Z_{GT2} + R_2 Z_{GT1})T_2 \\ (Z_{GT3} + R_1 Z_{GT2} - R_2 Z_{GT1})T_1 \\ \quad + (-Z_{GT1} - R_2 Z_{GT3} + R_3 Z_{GT2})T_3 \\ (-Z_{GT2} + R_1 Z_{GT3} - R_3 Z_{GT1})T_1 \\ \quad + (Z_{GT1} + R_2 Z_{GT3} - R_3 Z_{GT2})T_2 \end{Bmatrix} \tag{A6}$$

$$J'_{FXil} = \frac{\partial f'_i}{\partial \dot{X}_{Tl}} = \frac{M_{(i)} + \rho V_{wet} C_{A(i)(i)}}{\alpha \Delta t}\delta_{il} - N'_{D(i)}\delta_{il} - \frac{\partial N'_{D(i)}}{\partial \dot{X}_{Tl}}(V_i - \dot{X}'_{Ti} - \epsilon_{ijk} Z_{TBk} \dot{R}'_j) \tag{A7}$$

$$J'_{FRil} = \frac{\partial f'_i}{\partial \dot{R}_l} = \frac{M_{(i)} + \rho V_{wet} C_{A(i)(i)}}{\alpha \Delta t}\epsilon_{ijk} Z_{GTk}\delta_{jl} - \frac{\rho V_{wet} C_{A(i)(j+3)}}{\alpha \Delta t}\delta_{jl} - N'_{D(i)}\epsilon_{ijk} Z_{TBk}\delta_{jl} - \frac{\partial N'_{D(i)}}{\partial \dot{R}_l}(V_i - \dot{X}'_{Ti} - \epsilon_{ijk} Z_{TBk}\dot{R}'_j) \tag{A8}$$

$$J'_{FTil} = \frac{\partial f'_i}{\partial T_l} = -\delta_{il} \tag{A9}$$

$$\begin{aligned} J'_{HRil} = \frac{\partial h'_i}{\partial \dot{R}_l} &= \frac{I_{ij} + \rho V_{wet} C_{A(i+3)(j+3)}}{\alpha \Delta t}\delta_{jl} \\ &- \frac{\rho V_{wet} C_{A(i+3)(j)}}{\alpha \Delta t}\epsilon_{ijk} Z_{GTk}\delta_{jl} - \frac{\partial N'_{D(i)}}{\partial \dot{R}_l} W_{ij}\dot{X}'_{Tj} \\ &- \frac{\partial N'_{D(k)}}{\partial \dot{R}_l}\epsilon_{kjn} Z_{TBn} W_{ik}\dot{R}'_j - N'_{D(k)}\epsilon_{kjn} Z_{TBn} W_{ik}\delta_{jl} \\ &+ [Z_{GWi}F_{Wj} - Z_{GTi}T'_j + Z_{GBi}B_j\delta_{1j} - \rho g I_{wpj}\alpha\Delta t\delta_{jl} \\ &- \delta_{ij}(Z_{GWk}F_{Wk} - Z_{GTk}T_k + Z_{BGk}B_k\delta_{1k})]\alpha\Delta t\delta_{jl} \\ &- \epsilon_{ijk}\frac{\partial N'_{D(k)}}{\partial \dot{R}_l} Z_{GBj} V_k \end{aligned} \tag{A10}$$

$$\begin{aligned} J'_{HXil} = \frac{\partial h'_i}{\partial \dot{X}_{Tl}} &= \frac{\rho V_{wet} C_{A(i+3)(i)}}{\alpha \Delta t}\delta_{il} \\ &- \frac{\partial N'_{D(i)}}{\partial \dot{X}_{Tl}} W_{ij}\dot{X}'_{Tj} - N'_{D(k)} W_{ij}\delta_{jl} \\ &- \frac{\partial N'_{D(k)}}{\partial \dot{X}_{Tl}}\epsilon_{kjn} Z_{TBn} W_{ik}\dot{R}'_j \\ &- \epsilon_{ijk}\frac{\partial N'_{D(k)}}{\partial \dot{X}_{Tl}} Z_{GBj} V_k \end{aligned} \tag{A11}$$

$$J'_{HTil} = \frac{\partial h'_i}{\partial T_l} = [-Z_{GTi}\delta_{jl} - \delta_{ij}(-Z_{GTk}\delta_{kl})]\alpha\Delta t R'_j - \epsilon_{ijk} Z_{GTj}\delta_{kl} \tag{A12}$$