Simplified Earthquake Analysis of Structures with Foundation Uplift

By Anil K. Chopra,[1] M. ASCE and Solomon C.-S. Yim,[2] A. M. ASCE

Abstract: Simplified analysis procedures are developed to consider the beneficial effects of foundation-mat uplift in computing the earthquake response of structures, which respond essentially as single-degree-of-freedom systems in their fixed-base condition. These analysis procedures are presented for structures attached to a rigid foundation mat, which is supported on rigid foundation soil or flexible foundation soil modeled as two spring-damper elements. Winkler foundation with distributed spring-damper elements, or a viscoelastic half-space. In these analysis procedures, the maximum earthquake-induced base shear and deformation for an uplifting structure are computed directly from the earthquake response spectrum. It is demonstrated that the simplified analysis procedures provide results for the maximum base shear and deformation to a useful degree of accuracy for practical structural design.

Introduction

In dynamic analysis of response of buildings to earthquake ground motion, it is usual to postulate that the free-field ground motion is directly transmitted to the base of the structure without any modification. Implicit in this approach are the assumptions that the soil underlying the structure without any modification. Implicit in this approach are the assumptions that the soil underlying the structure is perfectly rigid and the structural foundation is firmly bonded to the soil. In reality, soils are not infinitely stiff and structural foundations are supported on the soil only through gravity forces.

During recent years, considerable research has been devoted to removing the approximation implied by the first of these assumptions (see Ref. 5 and its references). Rigorous analysis procedures have been developed to include the soil-structure interaction effects arising from soil flexibility and inertia in the earthquake response of structures, the principal effects of soil-structure interaction in structural response have been identified, and a simple, practical procedure has been developed to consider these effects in building design. However, much of the extensive work on soil-structure interaction was based on the second of the preceding assumptions, i.e., the foundation of the structure is firmly bonded to the soil.

Whether soil-structure interaction effects are considered or not, the earthquake-induced lateral forces on a building, computed by dynamic analysis under the assumption that the foundation and soil are firmly bonded, will, in many cases, produce a base overturning moment that exceeds the available overturning resistance due to gravity loads. The computed overload implies that a portion of the foundation mat or some

[1]Prof., Dept. of Civ. Engrg., Univ. of California, Berkeley, Calif.
[2]Asst. Research Engr., Dept of Civ. Engrg., Univ. of Calif., Berkeley, Calif.

Note.—Discussion open until September 1, 1985. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on November 30, 1983. This paper is part of the *Journal of Structural Engineering*, Vol. 111, No. 4, April, 1985. ©ASCE, ISSN 0733-9445/85/0004-0906/$01.00. Paper No. 19690.

of the individual column footings, as the case may be, would intermittently uplift during an earthquake. Several examples of towers and oil tanks uplifting from the underlying soil during Arvin Tehachapi (1952), Alaska (1964), and Imperial Valley (1979) earthquakes are cited in a recent work (4). Uplift of multistory building foundations has rarely been observed because the uplift is expected to be small and the foundation-soil interface is often inaccessible for observation.

Until a few years ago, the possibility of foundation uplift was not considered in the practical design of buildings because the design code forces were not large enough to initiate uplift. The situation changed after the San Fernando earthquake when a new code required that hospitals in California be designed for much larger forces. In some cases, the resulting base overturning moments exceeded the available overturning resistance due to gravity loads, and to prevent uplift structural, engineers anchored the foundation, a very expensive undertaking.

Recent analytical as well as shaking-table studies (2,3) of the complicated nonlinear response of buildings with their foundations permitted to uplift have demonstrated that there is no need to prevent foundation uplift because it generally has the effect of reducing the structural deformations and forces. In order to consider the beneficial effects of foundation uplift in computing the design forces and deformations for buildings, there is a need for a simple practical procedure to account for these effects. Such a procedure is developed in this paper for linear structures, which respond essentially as single-degree-of-freedom systems in their fixed base condition, attached to a rigid foundation mat, which is supported on rigid foundation soil, two spring-damper elements, Winkler foundation with distributed spring-damper elements, or a viscoelastic half-space.

Structures on Rigid Foundation Soil

System Considered.—The system consider first is shown in Fig. 1(*a*), consisting of an idealized representation of a one-story structure with its mass, m, concentrated at the roof level; height, h, above the massless foundation mat supported on nondeformable soil. The structure has linear properties with constant lateral stiffness, k, and lateral damping, c. The edges of the foundation mat are keyed into the nondeformable soil but not bonded to it. Thus, at each edge of the foundation mat, the soil can provide an upward reaction force to the foundation mat but not a downward pull.

The relation between the static moment, M, applied at the midpoint of the foundation mat, and the resulting mat rotation, θ, is shown in Fig. 2 for unbonded condition. The foundation mat undergoes no uplift or rotation as long as the overturning moment is below $M_c = mgb$, the maximum resisting moment due to the weight of the structure. If the overturning moment reaches M_c, the upward reaction force at one edge vanishes and that edge is in condition of incipient uplift from the supporting soil. Rotation about the other edge of the foundation mat can continue without any additional moment being developed. If the edges of the foundation mat were bonded to the undeformable soil, unlimited

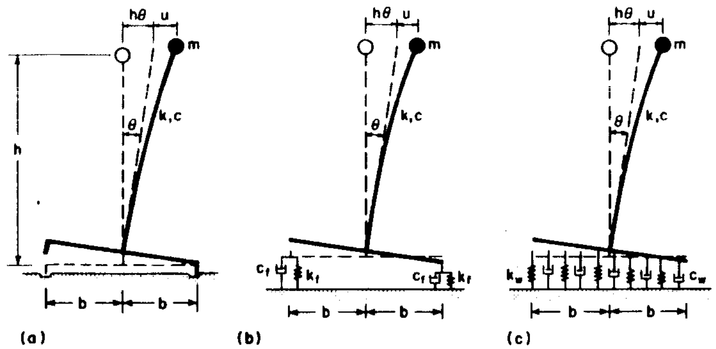

FIG. 1.—Structure Supported on Three Different Foundation Soil Idealizations: (*a*) Rigid Foundation; (*b*) Two-Element Foundation; (*c*) Winkler Foundation

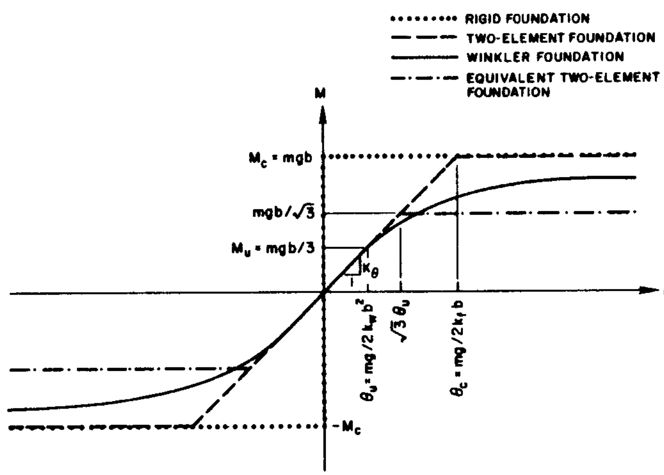

FIG. 2.—Moment-Rotation Relations for Unbonded Foundation Mat Supported on Different Foundation Soil Idealizations

overturning moment can be developed (if p-δ effects are ignored) without any foundation rotation.

The base excitation is specified by the horizontal ground motion with displacement, $u_g(t)$, and acceleration, $\ddot{u}_g(t)$. Under the influence of this excitation, the displaced configuration of the structure at any instant of time can be defined by the deformation, $u(t)$, and foundation-mat rotation, $\theta(t)$.

Equations of Motion.—The equations governing the motion of the structure for two conditions of contact between the foundation mat and supporting soil—no uplift and with uplift—are presented along with the criteria for transition between the two contact conditions. These equations are presented under the assumption that the structure and excitation are such that the amplitudes of the resulting displacement and

rotation responses are small, so that sin θ and cos θ may be approximated by θ and 1, respectively, and $mg/kh << 1$, so that p-δ effects are negligible (6).

1. No Uplift.—When both edges of the foundation mat are in contact with the supporting soil, the motion of the structure is governed by the standard differential equation for a single-degree-of-freedom system:

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = -\ddot{u}_g(t) \quad (1)$$

in which $\omega = \sqrt{k/m}$ = the natural vibration frequency of the structure; and $\xi = c/2m\omega$ = the damping ratio.

2. With Uplift.—When one edge of the foundation mat is uplifted from the supporting soil, the governing equations can be derived by considering the lateral equilibrium of forces acting on the structural mass and moment equilibrium of forces acting on the entire system, and expressed as

$$\ddot{u} + h\ddot{\theta} + 2\xi\omega\dot{u} + \omega^2 u = -\ddot{u}_g(t) \quad (2a)$$

$$\ddot{u} + \frac{r^2}{h^2}(h\ddot{\theta}) = -\ddot{u}_g(t) \mp g\frac{b}{h} \quad (2b)$$

in which the upper and lower algebraic signs correspond, respectively, to uplift of the left and right edge of the foundation mat; and $r = \sqrt{h^2 + b^2}$ = the diagonal distance from the mass to an edge of the footing measured under undeformed conditions.

Eliminating $(h\ddot{\theta})$ algebraically from Eqs. 2*a*–*b* leads to

$$\ddot{u} + 2\zeta\lambda\dot{u} + \lambda^2 u = -\left[\ddot{u}_g(t) \mp g\frac{h}{b}\right] \quad (3)$$

in which $\lambda = \omega\left(\frac{r}{b}\right)$ (4*a*)

$$\zeta = \xi\left(\frac{r}{b}\right) \quad (4b)$$

are the natural frequency and damping ratio of the uplifted system. For slender structures the frequency, λ, is approximately equal to $\omega(h/b)$, which is much higher than the fixed-based frequency, ω. Similarly, ζ is approximately $\xi(h/b)$, which is much higher than the fixed-base damping ratio, ξ.

3. Initiation of Uplift.—The foundation mat would begin to rock about one edge, causing uplift of the other edge, when the overturning moment, M, exceeds the available resisting moment, M_c, due to gravity, i.e.

$$M > mgb \quad (5a)$$

or $mh|\ddot{u} + \ddot{u}_g(t)| > mgb$ (5*b*)

Rocking takes place about the left edge if the acceleration $\ddot{u} + \ddot{u}_g(t)$ of the mass is positive and about the right edge if the acceleration is negative.

For undamped structures, the overturning moment $M = khu_c$ in which u_c = the structural deformation at the onset of foundation-mat uplift. Substituting this relation for M into Eq. 5a leads to

$$u_c = \frac{mg}{k}\frac{b}{h} = \frac{g}{\omega^2}\frac{b}{h} \qquad (6)$$

Thus for undamped structures, the uplift criterion of Eq. 5 simplifies to $u > u_c$.

The critical deformation, u_c, may also be viewed as the structural deformation at incipient uplift of the foundation mat under the action of gradually increasing lateral force applied at the mass, m. Furthermore, when subjected to static lateral force, the structural deformation cannot increase beyond u_c and the associated base shear is

$$V_c = mg\frac{b}{h} \qquad (7)$$

4. Recontact after Uplift.—As mentioned in Ref. 3, it is appropriate to idealize the foundation mat-soil impact as a perfectly inelastic collision (a dull thud, without bouncing of the foundation mat), which completely dissipates the vertical velocity of the structural mass, so that

$$\theta = \dot{\theta} = 0 \qquad (8a)$$

Conservation of horizontal momentum of the structural mass requires that

$$(\dot{u} + h\dot{\theta})|_{\text{before impact}} = \dot{u}|_{\text{after impact}} \qquad (8b)$$

Analysis Procedure.—The motion of the structure in the absence of foundation uplift is governed by a linear differential equation (Eq. 1) and, in the presence of foundation uplift, by a pair of linear differential equations (Eq. 2). The solution of each set of governing equations can be expressed as the sum of a complementary solution with arbitrary constants depending on the initial conditions and a particular solution corresponding to the given excitation (3). By proper choice of these arbitrary constants, it is possible to match the two solutions (in the absence and presence of foundation uplift) at the transition times when the foundation mat begins and ceases to uplift. Criteria to determine these transition times, and the displacements and velocities after a transition are described by Eqs. 5 or 8 as appropriate.

While the solution procedure outlined earlier could be implemented analytically if the excitation was simple enough to permit a closed-form particular solution, numerical implementation would be necessary for complicated earthquake-like excitations. The results to be presented later were obtained by numerically integrating Eqs. 1 and 2, sequentially, subject to the transition criteria of Eqs. 5 and 8, using an implicit Newmark method with linear variation of acceleration in each time step, Δt. Within each Δt, the instant of transition when the foundation mat begins or ceases to uplift is obtained by an iterative process. The time step, Δt, was chosen to be shorter than one-tenth of the smallest vibration period, $2\pi/\lambda$, and small enough to provide accurate results.

Free Vibration Response.—Numerical results for the response of an

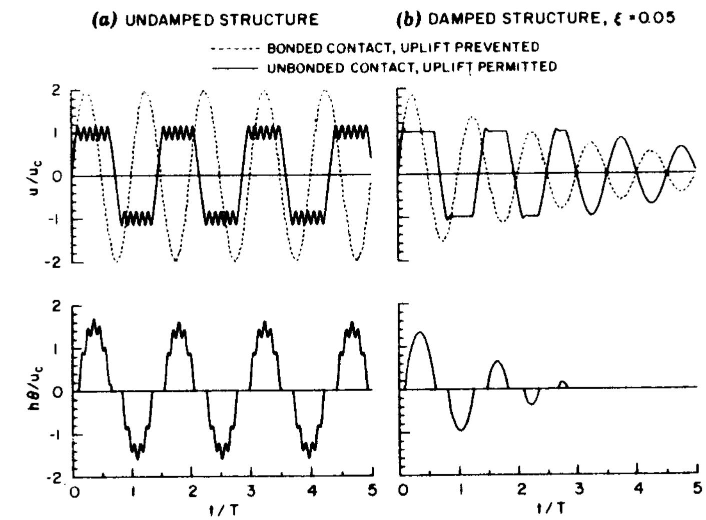

FIG. 3.—Response of Structures (h/b = 10) to Initial Velocity for Two Conditions of Contact Between Foundation Mat and Supporting Soil: (a) Undamped Structure; (b) Damped Structure, ξ = 0.05

undamped structure to a prescribed initial velocity are presented in Fig. 3(a) for two conditions of contact between the foundation mat and the supporting soil: (1) Bounded contact preventing uplift; and (2) unbonded contact only through gravity with uplift permitted. The response quantities presented as a function of time are deformation, u, and height times foundation-mat rotation, $h\theta$, both normalized with respect to the critical deformation, u_c. If the foundation mat is bonded to the supporting soil, the structural response is the familiar simple harmonic motion with vibration frequency, ω, the natural vibration frequency of the single-degree-of-freedom system. If the foundation mat is not bonded to the supporting soil and uplift is permitted, the footing rocks about the right edge and the left edge uplifts as soon as u reaches u_c. Solution of Eq. 3 with $\ddot{u}_g(t) = 0$ would indicate that, during foundation uplift, the deformation is given by a constant part = $(h^2/r^2)u_c$, which for a wide range of values for the slenderness h/b is close to u_c (Eq. 6), plus an oscillating component at frequency, λ (Eq. 4a), which for slender structures is much higher than ω, as mentioned earlier. For the example structure, h/b = 10 and λ is approximately ten times ω. After a few cycles of the high-frequency vibration in uplifted condition, contact at both edges of the foundation mat is re-established. The structure passes through its at-rest position until uplift of the right edge is initiated and the footing begins to rock about the left edge resulting in a few cycles of vibration at frequency, λ, until contact at both edges of the foundation mat is established once again. This response behavior is repeated during every cycle of free vibration as seen in Fig. 3(a). Since the foundation-mat-soil impact was assumed to be a perfectly inelastic collision, the foundation-mat does not bounce, and, as seen Fig. 3(a), each impact is followed by a period of time during which both edges of the foundation mat remain in contact with the underlying soil.

The response of the same structure but now with damping, such that the damping ratio ξ = 5%, to the same initial velocity are presented in Fig. 3(*b*). Because of structural damping, the tendency of the foundation mat to uplift is reduced, resulting in the amount and duration of uplift decreasing with each vibration cycle, and, after a few cycles, the foundation mat remains in continuous contact with the soil. If the foundation mat is bonded to the suporting soil, the structural response decays at a rate controlled by the damping ratio, ξ, for the structure. If the foundation mat is not bonded to the supporting soil and uplift is permitted, the vibration mode of the uplifted system, which contributes the high-frequency oscillatory deformation, has a modal damping ζ (Eq. 4*b*), which for slender structures, is much larger than the damping ratio, ξ, for the structure, as mentioned earlier. For this example structure, $h/b = 10$ and ζ is approximately 50%, i.e., ten times ξ. Because of high damping, the vibratory response at frequency λ is essentially eliminated as seen in Fig. 3(*b*). During each episode of response with foundation-mat uplift, the deformation is essentially constant but for the one (could be more depending on system properties) spike at the beginning of the episode.

Earthquake Response.—The effects of transient foundation uplift on the earthquake response of a structure are displayed in Fig. 4 for an example structure, subjected to the north-south component of El Centro, 1940, ground motion. If the foundation mat is bonded to the supporting soil, the structure responds as a SDOF system with natural vibration period = 1 sec and damping ratio = 5%. The response behavior is strikingly different if the foundation mat is supported on the underlying soil only through gravity, thus permitting uplift. During the initial phase of ground shaking, both edges of the foundation mat remain in contact with the underlying soil. As the ground motion intensity builds up and the deformation exceeds the critical deformation, u_c (the precise uplift criterion is slightly different for damped systems as noted earlier), the foundation mat rocks alternately about its two edges in a vibration cycle, with the amount and duration of uplift varying from cycle to cycle. As the intensity of the ground motion decays, the deformation response remains below the critical deformation, u_c, and the foundation mat remains in contact with the supporting soil unless a later burst of strong ground motion causes increased response and foundation mat uplift. The first six seconds of the response history of Fig. 4, for a structure with damping ratio ξ = 5%, is presented in Fig. 5(*b*), with the corresponding plot for an undamped structure. For reasons presented in greater detail in the preceding section, the high-frequency (λ) vibratory response, which exists for undamped structures during each episode of response with foundation-mat uplift, is essentially eliminated because it is heavily damped; ζ = 50%. Consequently, as seen in Figs. 4 and 5(*b*), the deformation response is essentially constant during each episode of response with foundation mat uplift, except for the one spike at the beginning of the episode [Figs. 4 and 5(*b*)]. However, the amplitude and number of spikes at the beginning of such a response episode depends on the system properties, as can be observed by comparing Figs. 5(*b–c*). It is apparent from Fig. 4 that the maximum deformation of this example

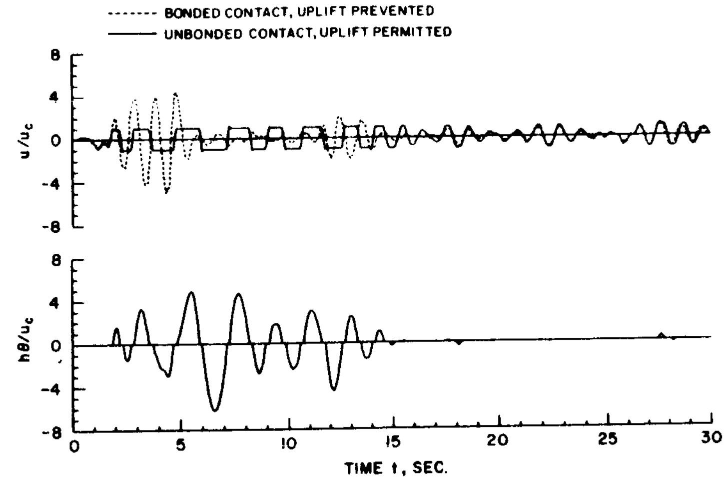

FIG. 4.—Response of Structure (T = 1 sec; ξ = 0.05; h/b = 10) to El Centro Ground Motion for Two Conditions of Contact Between Foundation Mat and Supporting Soil

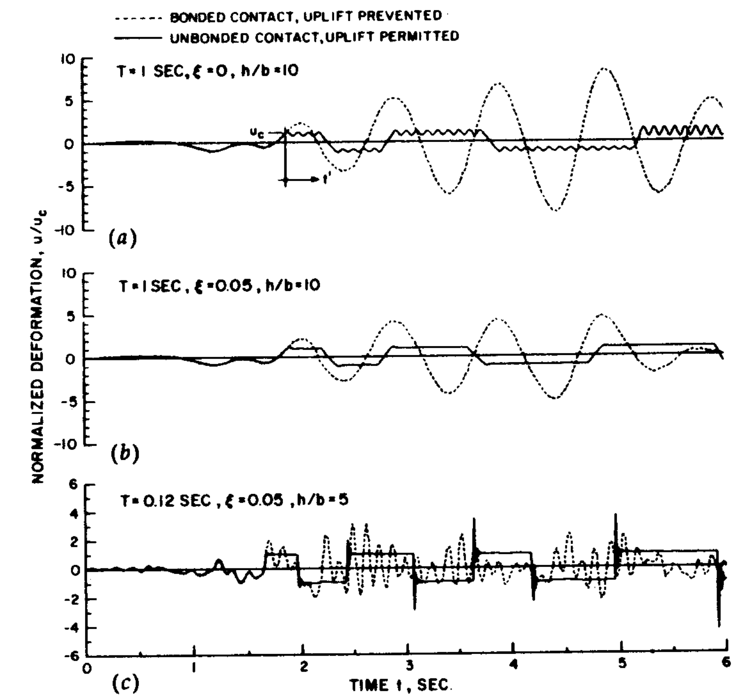

FIG. 5.—Response of Three Structures to El Centro Ground Motion for Two Conditions of Contact Between Foundation Mat and Supporting Soil: (*a*) T = 1 sec, ξ = 0, h/b = 10; (*b*) T = 1 sec, ξ = 0.05, h/b = 10; (*c*) T = 0.12 sec, ξ = 0.05, h/b = 5

structure subjected to the selected ground motion is significantly reduced because of foundation-mat uplift. However, as seen in Fig. 5(*c*), it is quite possible that the maximum deformation may increase in spite of foundation-mat uplift. The increased response usually appears in the form of a single spike without complete reversal of deformation at the beginning of some of the response episodes with foundation-mat uplift, which would be less damaging than the same response amplitude but with deformation reversal.

From the practical point of view, it would be most useful to know the effect of foundation-mat uplift on the maximum base shear, V_{max}, caused by the earthquake. To this end, the base shear coefficient $\mathcal{V}_{max} = V_{max}/mg$ is plotted as a function of the fixed-base natural vibration period of the structure $T = 2\pi/\omega$. The response spectra presented in Fig. 6 are for four values of the slenderness ratio, h/b. Also shown is the critical base shear coefficient, $\mathcal{V}_c = V_c/mg = b/h$, which varies inversely with slenderness ratio, h/b. If the foundation mat is bonded to the supporting soil and it cannot uplift, the response spectrum is independent of the slenderness ratio, and it is simply the standard pseudo-acceleration spectrum, normalized with respect to gravitational acceleration. The base shear developed in structures with relatively long vibration periods is below the critical value and the foundation mat, even if it rests on the supporting soil only through gravity, does not uplift throughout the earthquake. If foundation-mat uplift is prevented, the maximum base shear at some vibration periods may exceed the critical value. For the selected system parameters and ground motion, Fig. 6 indicates that the critical value is exceeded for all vibration periods shorter than where the standard pseudo-acceleration spectrum first attains the critical value. If the foundation mat of such a structure rests on the supporting soil only through gravity, uplift of the foundation mat occurs during the earthquake and this generally has the effect of reducing the base shear. Because the critical base shear varies inversely as the slenderness ratio, h/b, the foundation mat of a slender structure has a greater tendency to uplift resulting in greater reductions in base shear. However the base shear is not reduced to as low as the critical value, V_c, based on static considerations, because, although foundation-mat uplift is initiated at the time instant the base shear reaches V_c, this value is exceeded by the instantaneous (dynamic) critical value, $m(g + |b\ddot{\theta}|)b/h$, resulting in maximum base shear larger than V_c. As can be observed by comparing Figs. 5(*b–c*), this tendency is more pronounced for structures with shorter vibration periods.

Although the maximum base shear is reduced because of foundation-mat uplift for structures over a wide range of fixed-base vibration period, very short period structures may be exceptions to this behavior. Their response, as seen in Figs. 5(*c*) and 6, may increase because of foundation mat uplift. However, these vibration periods are unrealistically short for most structures with slenderness ratios in the range 5–20, which are representative of medium-rise to high-rise structures with longer vibration periods.

Simplified Analysis Procedure.—The response behavior of structures, excluding those with very short fixed-base vibration period, is simple when their foundation mat is permitted to uplift even though the system

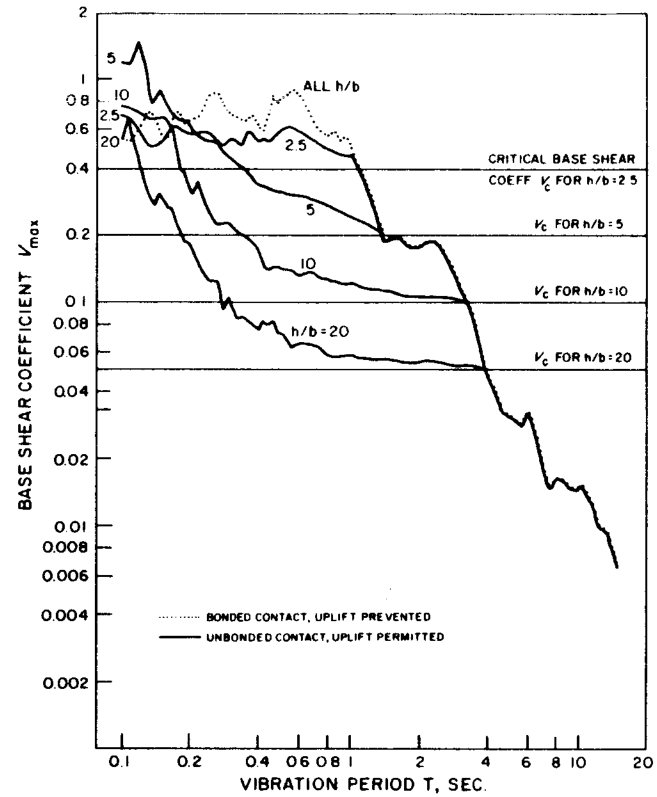

FIG. 6.—Response Spectra for Structures (ξ = 0.05, h/b = 2.5, 5, 10 or 20) on Rigid Foundation Soil Subjected to El Centro Ground Motion for Two Conditions of Contact Between Foundation Mat and Supporting Soil

is nonlinear. In a sense, the behavior seems to be even more simple than that of the corresponding linear system—the structure with its foundation mat bonded to the supporting soil. Whereas the maximum deformation of the linear system may occur after a few cycles of response building up during an earthquake, as seen in Fig. 4, the maximum deformation (or close to it) occurs soon after the foundation mat uplifts for the first time. In fact, as shown in Fig. 5, it occurs at the first peak of the high frequency (λ) vibratory response (Eq. 4*a*). This high frequency oscillation is undamped if the structure has no damping but is highly damped with damping ratio, ζ (Eq. 4*b*), if the structure has some damping and the vibratory response is essentially eliminated after the first peak, which seems to be influenced little by the ground acceleration, $\ddot{u}_g(t)$. It therefore seems possible to exploit these features of the response behavior in developing a simpilifed procedure to compute the maximum deformation of a structure with its foundation mat permitted to uplift.

Because the deformation response of a structure during the time that its foundation mat is uplifted does not seem to depend significantly on the ground motion $\ddot{u}_g(t)$, it may be possible to obtain a good estimate of this response from Eq. 3 without the ground acceleration:

$$\ddot{u} + 2\zeta\lambda\dot{u} + \lambda^2 u = \pm\frac{gh}{b} \quad (9)$$

For undamped structures ($\xi = 0$, $\zeta = 0$) the solution of Eq. 9 can be written as

$$u(t') = u_c\frac{h^2}{r^2} + u_c\frac{b^2}{r^2}\cos\lambda t' + \frac{\dot{u}_c}{\lambda}\sin\lambda t' \quad (10)$$

in which time t' is measured from the onset of foundation rocking (uplift) as shown in Fig. 5(a); u_c and $\dot{u}_c$ = the structural deformation and velocity at the instant the footing begins to rock. The critical deformation, u_c, is given by Eq. 6; and the velocity $\dot{u}_c$ may be determined by recognizing that this is also the velocity of the corresponding linear system, with bonded foundation mat, at that instant of time [Fig. 5(a)]. The velocity, $\dot{u}_c$, can be determined by using the fact that any one cycle of deformation of this linear system is essentially sinusoidal with amplitude, u_o, and frequency, ω, with the amplitude and phase of the response varying slowly from cycle to cycle during an earthquake, a result known from random vibration theory (1). Thus

$$\dot{u}_c = \omega\sqrt{u_o^2 - u_c^2} \quad (11)$$

Substituting Eq. 11 into 10, the maximum value of $u(t)$ can be expressed as

$$u_{max} = u_c\left\{\frac{h^2}{r^2} + \sqrt{\frac{b^4}{r^4} + \frac{b^2}{r^2}\left[\left(\frac{u_o}{u_c}\right)^2 - 1\right]}\right\} \quad (12)$$

a result originally derived by Meek (3) in a study restricted to undamped systems.

For damped structures, the solution of Eq. 9 is

$$u(t') = u_c\frac{h^2}{r^2} + e^{-\zeta\lambda t'}\left(u_c\frac{b^2}{r^2}\cos\lambda t' + \frac{\dot{u}_c}{\lambda}\sin\lambda t'\right) \quad (13)$$

Just as in the undamped case, the structural deformation, u_c, at the instant the foundation mat begins to rock is the same as the critical deformation, u_c, given by Eq. 6; and the velocity, $\dot{u}_c$, at the instant the mat begins to rock is approximately equal to the velocity of the corresponding undamped linear system, with bonded foundation mat. Any one cycle of deformation of this linear system is essentially sinusoidal with frequency, ω, and amplitude = $e^{\xi\pi/2}u_o$ in which u_o = the amplitude of the first response peak of the damped system. Thus, Eq. 11 becomes

$$\dot{u}_c = \omega\sqrt{e^{\xi\pi}u_o^2 - u_c^2} \quad (14)$$

Substituting Eq. 14 into 13, after some simplification the maximum value of $u(t')$ can be expressed as

$$u_{max} = u_c\left\{\frac{h^2}{r^2} + e^{-\zeta\phi}\sqrt{\frac{b^4}{r^4} + \frac{b^2}{r^2}\left[\left(\frac{u_o}{u_c}\right)^2 e^{\xi\pi} - 1\right]}\right\} \quad (15a)$$

where $$\phi = \frac{\pi}{2} - \tan^{-1}\left\{\frac{b}{r}\left[\left(\frac{u_o}{u_c}\right)^2 e^{\xi\pi} - 1\right]^{-1/2}\right\} \quad (15b)$$

Note that Eq. 12, the corresponding result for undamped systems, is a special case of Eq. 15.

Thus, the maximum earthquake induced deformation of a structure with its foundation mat permitted to uplift can be conveniently computed from Eq. 12 or 15 provided the first peak, u_o, after initiation of uplift in the deformation response of the corresponding linear system (with bonded foundation mat) is known. Whereas this peak can be computed by solving Eq. 1, it is not readily available. However, the largest of all peaks in the response history of a linear system is simply equal to

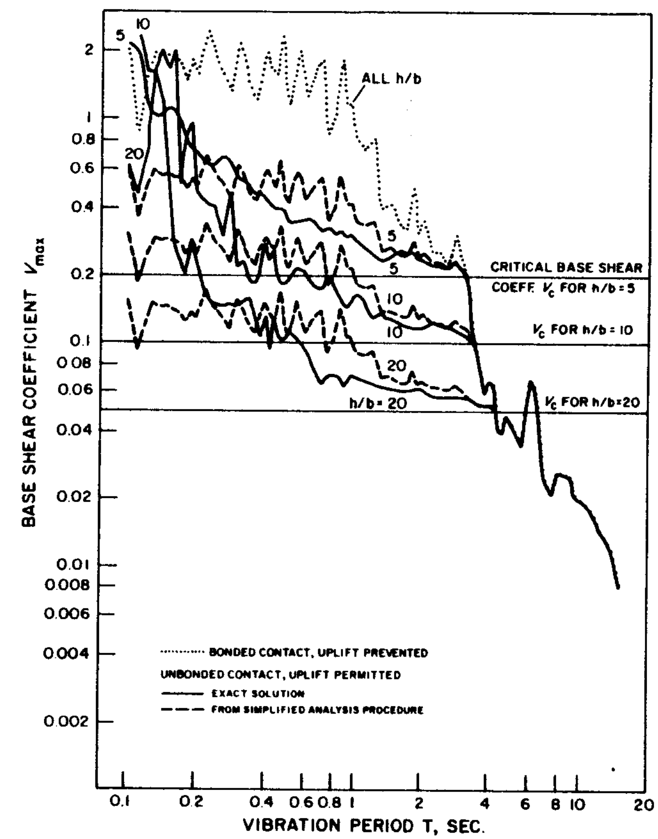

FIG. 7.—Comparison of Base Shear Response Spectra, Computed by Exact and Simplified Analysis Procedures, for Undamped Structures (h/b = 5, 10 or 20) on Rigid Foundation Soil Subjected to El Centro Ground Motion

$S_d(\omega, \xi)$, the ordinate of the deformation response spectrum for the earthquake corresponding to vibration frequency, ω, and damping ratio, ξ. Replacing u_o by S_d, Eq. 15 provides an expression for the maximum deformation, u_{max}, of an uplifting system in terms of the deformation response spectrum for linear systems. The corresponding base shear, $V_{max} = ku_{max}$, after utilizing Eqs. 6 and 7, can be expressed as

$$V_{max} = V_c\left\{\frac{h^2}{r^2} + e^{-\zeta\phi}\sqrt{\frac{b^4}{r^4} + \frac{b^2}{r^2}\left[\left(\frac{S_a}{g}\right)^2\left(\frac{h}{b}\right)^2 e^{\xi\pi} - 1\right]}\right\} \quad (16a)$$

in which
$$\phi = \frac{\pi}{2} - \tan^{1}\left\{\frac{b}{r}\left[\left(\frac{S_a}{g}\right)^2\left(\frac{h}{b}\right)^2 e^{\xi\pi} - 1\right]^{-1/2}\right\} \quad (16b)$$

and $S_a(\omega, \xi) = \omega^2 S_d(\omega, \xi)$ = the ordinate of the pseudo-acceleration response spectrum.

For slender structures, Eq. 16 can be simplified to

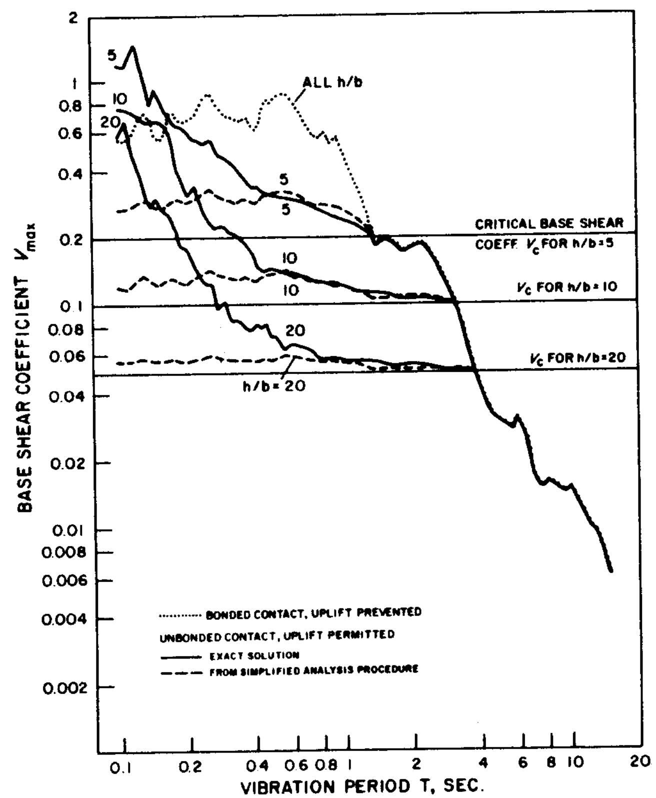

FIG. 8.—Comparison of Base Shear Response Spectra, Computed by Exact and Simplified Analysis Procedures, for Damped Structures (ξ = 0.05; h/b = 5, 10 or 20) on Rigid Foundation Soil Subjected to El Centro Ground Motion

$$V_{max} = V_c\left\{1 + \frac{b}{r}e^{-\zeta\pi/2}\sqrt{\left(\frac{S_a}{g}\right)^2\left(\frac{h}{b}\right)^2 - 1}\right\} \quad (17)$$

The errors introduced in replacing Eq. 16 by Eq. 17 are generally less than approximately 5% for structures with $h/b \geq 5$.

The approximate value of the maximum base shear coefficient $V_{max} = V_{max}/mg$ computed from the simplified analysis procedure, Eq. 16, is plotted in Fig. 7 for undamped systems and in Fig. 8 for damped systems, with the maximum response obtained from response history computed by solving the equations of motion, Eqs. 1 and 2, subject to the transition criteria of Eq. 5 and 8. For a wide range of values for the fixed-base natural vibration period, $T = 2\pi/\omega$ of the structure, the simplified analysis procedure provides excellent results, especially for damped structures, because, consistent with the assumptions underlying the simplified procedure, the maximum deformation of the linear system is also attained within a very few oscillations after the uplifting system attains its maximum deformation (Fig. 4). Results from the simplified analysis procedure deteriorate for short-period structures because, contrary to the underlying assumptions, the maximum response of such a structure can occur after several response episodes with uplift of the foundation-mat, at a time instant that essentially has no relation to the instant the maximum deformation of the linear system occurs [Fig. 5(c)]. That the simplified analysis procedure fails to accurately predict the response of short period structures is not a serious limitation because such structures are usually squatty and not likely to uplift during earthquakes.

Structures on Flexible Foundation Soil

Systems Considered.—The two systems considered next are shown in Figs. 1(*b–c*) wherein the idealized one-story structure is now supported through a massless foundation mat resting on flexible soil. In one case, the flexible soil is represented by two spring-damper elements, one at each edge of the foundation mat, and, in the other, by Winkler model with spring-damper elements distributed over the entire width of the mat. These two foundation models have been considered in detail elsewhere (6).

The relation between the static moment, M, applied at the midpoint of the foundation mat supported on flexible soil and the resulting mat rotation, θ, is shown in Fig. 2. Flexibility of the foundation soil leads to finite slope, with rocking stiffness, K_θ, in the static moment-rotation relation for the foundation mat assumed to be bonded to the underlying soil. If the mat is not bonded to the underlying soil, the M-θ relation is linear, implying constant rocking stiffness, until one edge of the foundation mat uplifts from the supporting elements. The foundation mat supported on a two-element model uplifts when M reaches $M_c = mgb$; thereafter no additional moment can be developed. Uplift of the foundation mat supported on Winkler foundation is initiated when the overturning moment reaches $M_u = mgb/3$. Thereafter, the rocking stiffness decreases monotonically with increasing θ, which implies an expanding foundation-mat width over which uplift occurs; and the M-θ curve asymptotically approaches the critical moment, $M_c = mgb$, corresponding to the physically unrealizable condition of uplift of the entire foundation mat from the supporting elements except for one edge. Note that the maximum overturning moment that can be sustained under static conditions, $M_c = mgb$, is independent of whether the soil is idealized as flexible or rigid and of the idealization used to represent soil flexibility. The corresponding value of base shear, $V_c = mg(b/h)$.

If the foundation mat is bonded to the supporting elements, the equations of motion for the structure supported on Winkler foundation are identical to those for the same structure on a two-element foundation with the following properties: $k_f = bk_w$, $c_f = bc_w$ and half-base width = $b/\sqrt{3}$. This two-element foundation is exactly equivalent to the Winkler foundation if uplift is not permitted. If the foundation mat is not bonded to the supporting elements and it is permitted to uplift, the M-θ relations for the Winkler foundation and this equivalent two-element system are compared in Fig. 2. Obviously the M-θ relation is linear with the same rocking stiffness for the two systems until M reaches M_u, when one edge of the foundation mat on the Winkler foundation is at incipient uplift. For larger overturning moments, the M-θ relation for the two systems are different, as shown in Fig. 2, with the equivalent two-element system deteriorating for the larger rotation angles that would develop if the ground motion is intense enough to cause considerable uplift.

In the preceding section of this paper, impact between the foundation mat and rigid supporting soil was idealized as a perfectly inelastic collision with the vertical velocity and associated kinetic energy of the structural mass dissipated instantaneously. However, when foundation flexibility is considered, the energy is gradually dissipated through the viscous damping mechanism of the foundation. But if the foundation springs are relatively stiff, the typical case for buildings and soil, the energy dissipation occurs within a very short duration. Thus, the two types of energy dissipating mechanisms would be expected to similarly affect structural response.

Analysis Procedure.—The analysis of the structure-foundation systems shown in Figs. 1(b–c) with the foundation mat permitted to uplift has been the subject of recent studies (4,6). The foundation stiffness and damping parameters in the nonlinear equations of motion for the structure depend on whether the foundation mat is in contact with the supporting system at one or both edges, and also on the extent of uplift in case of the Winkler foundation (6). However, a linear system corresponding to any instantaneous contact condition of the foundation mat can be defined. This linear system has three degrees of freedom and the natural frequency of the third mode is an order of magnitude higher than the second mode frequency, and it tends to infinity as the foundation-mat mass approaches zero. Because the contribution of the high-frequency mode to the response is negligible, the Rayleigh-Ritz concept was employed to eliminate this mode from the analysis (6). It was then possible to employ a much larger time-step than would otherwise be practical, resulting in considerable savings in computational effort. Response spectra were developed for a range of the important system parameters to investigate the effects of foundation mat uplift on structural response and how these effects depend on the various system parameters.

This complicated analysis procedure is not convenient for practical application. Thus, the objective here is to seek a simplified analysis procedure, which although approximate is accurate enough, to consider foundation mat uplift in the computation of maximum structural response. To this end, we attempt to relate the maximum response of a flexibly supported system to that for a rigidly supported system, which is much simpler to analyze as seen in the preceding section.

Simplified Analysis Procedure.—Assuming that the foundation mat is bonded to the supporting elements, the equation governing the coupled structural deformation and foundation rocking motions of the systems of Figs. 1(b–c) with massless foundation mat subjected to horizontal ground motion can be expressed as (see Appendix II)

$$\left[1 + \frac{kh^2}{K_\theta}\right]^2 m\ddot{u} + \left[c + \frac{C_\theta}{h^2}\left(\frac{kh^2}{K_\theta}\right)^2\right]\dot{u} + k\left[1 + \frac{kh^2}{K_\theta}\right]u = -m\left[1 + \frac{kh^2}{K_\theta}\right]\ddot{u}_g(t) \quad \text{(18)}$$

in which K_θ and C_θ = the rocking stiffness and damping of the foundation mat, defined by

$$K_\theta = \begin{cases} 2k_f b^2 & \text{for two-element foundation} \\ \dfrac{2k_w b^3}{3} & \text{for Winkler foundation} \end{cases} \quad \text{(19a)}$$

$$C_\theta = \begin{cases} 2c_f b^2 & \text{for two-element foundation} \\ \dfrac{2c_w b^3}{3} & \text{for Winkler foundation} \end{cases} \quad \text{(19b)}$$

Note that the rocking stiffness and damping for the equivalent two-element system defined earlier are identical to those for the Winkler foundation defined in Eq. 19. Although Eq. 18 is exact for systems without damping in the structure or foundation, as seen in Appendix II, it is only approximately valid for damped systems.

Eq. 18 can be rewritten as

$$\ddot{u} + 2\tilde{\xi}\tilde{\omega}\dot{u} + \tilde{\omega}^2 u = -\left(\frac{\tilde{\omega}}{\omega}\right)^2 \ddot{u}_g(t) \quad \text{(20)}$$

in which the effective natural frequency, $\tilde{\omega}$, and associated natural period, $\tilde{T}$, of the flexibly supported system are given by

$$\tilde{\omega} = \frac{\omega}{\sqrt{1 + \dfrac{kh^2}{K_\theta}}} \quad \text{and} \quad \tilde{T} = T\sqrt{1 + \frac{kh^2}{K_\theta}} \quad \text{(21)}$$

and the effective damping ratio, $\tilde{\xi}$, by

$$\tilde{\xi} = \frac{\xi}{\left(\frac{\tilde{T}}{T}\right)^3} + \tilde{\xi}_o \quad \text{(22)}$$

The effective damping of the flexibly supported structure consists of two terms, representing the contribution of structural and foundation damping, respectively, with the latter given by

$$\tilde{\xi}_o = \frac{1}{2m\omega}\frac{C_\theta}{h^2}\frac{\left(\frac{\tilde{T}}{T}\right)^2 - 1}{\left(\frac{\tilde{T}}{T}\right)^3} \quad \text{(23)}$$

Flexibility of the foundation supporting system has the effect of lengthening the vibration period, resulting in $\tilde{T}$ greater than T and reducing the effectiveness of the structural damping. This reduction may be quite significant when $\tilde{T}/T$ is large. However, $\tilde{\xi}$ may be greater than, equal to, or smaller than ξ. Eqs. 21–23 for the effective vibration period and damping of the flexibly supported structure are special cases of the results obtained for structures supported on foundation mat bonded to a viscoelastic half-space (5).

The maximum value of the deformation, $u(t)$, governed by Eq. 20 can be expressed in terms of the deformation response spectrum, $S_d(T,\xi)$, for the earthquake ground motion:

$$u_{\max} = \left(\frac{\tilde{\omega}}{\omega}\right)^2 \tilde{S}_d = \frac{\tilde{S}_d}{\left(\frac{\tilde{T}}{T}\right)^2} \quad \text{(24)}$$

in which $\tilde{S}_d = S_d(\tilde{T},\tilde{\xi})$. By utilizing Eq. 24 and the definition of pseudo-acceleration spectrum, $S_a = \omega^2 S_d$, the maximum base shear, $V_{\max} = ku_{\max}$, can be expressed as follows:

$$V_{\max} = mg\left(\frac{\tilde{S}_a}{g}\right) \quad \text{(25)}$$

in which $\tilde{S}_a = S_a(\tilde{T},\tilde{\xi})$.

The responses of flexibly and rigidly supported systems are related through Eqs. 24 and 25 because both systems are linear if the foundation mat is firmly bonded to the supporting soil. These equations would provide exact results for the response of flexibly supported structures with no damping in the structure or the supporting system, but only approximate results for damped systems because, as seen in Appendix II, Eq. 18 is exactly valid for undamped systems but not for damped systems. The approximate results obtained from these equations for damped systems are compared in Figs. 9 and 10 with the results obtained by the exact analysis procedure (6) for structures supported on two-element or Winkler foundations, and the agreement is seen to be essentially perfect. Thus, if the foundation mat is bonded to the supporting soil and it is not permitted to uplift, the maximum response of a structure with fixed-

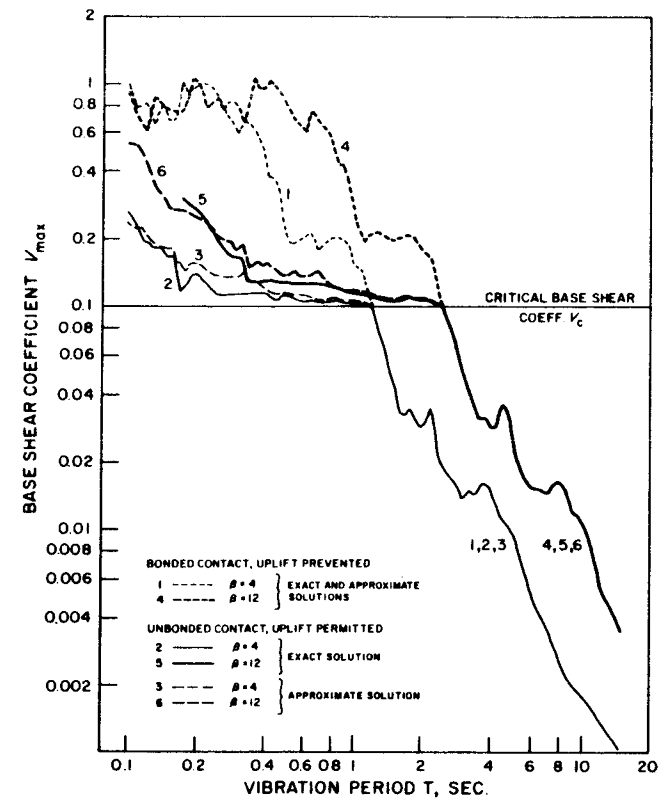

FIG. 9.—Comparison of Base Shear Response Spectra, Computed by Exact and Approximate (Eq. 25) Analysis Procedures, for Structures on Two-Element Foundation Subjected to El Centro Ground Motion (h/b = 10; $\beta \equiv \omega/\omega \equiv \sqrt{2k_f/m} \div \sqrt{k/m}$ = 4 or 12; ξ = 0.05; $\xi_v \equiv 2c_f/2m\omega_v$ = 0.4)

base vibration period, T, and damping ratio, ξ, supported on flexible foundation soil can be determined almost exactly by analyzing the response of a structure with vibration period, $\tilde{T}$, and damping ratio, $\tilde{\xi}$, on rigid soil and using Eqs. 24 and 25.

If the mat is not bonded to the supporting elements and it uplifts intermittently during intense earthquake motion, the resulting response will be nonlinear, and it does not seem possible to derive a rational analytical relationship between the responses of flexibly and rigidly supported systems. Now, considering that the governing equations for rigidly supported and flexibly supported (on two-element foundation) structures prior to uplift (Eqs. 1 and 20) are similar, that uplift is initiated at the same critical deformation for both systems, that the response histories of rigidly supported systems (presented in this paper) and of flexibly supported systems presented elsewhere (6) have similar characteristics, and that (at least for long-period structures) the maximum response occurs at the first high-frequency peak after uplift (Fig. 5), it seems that

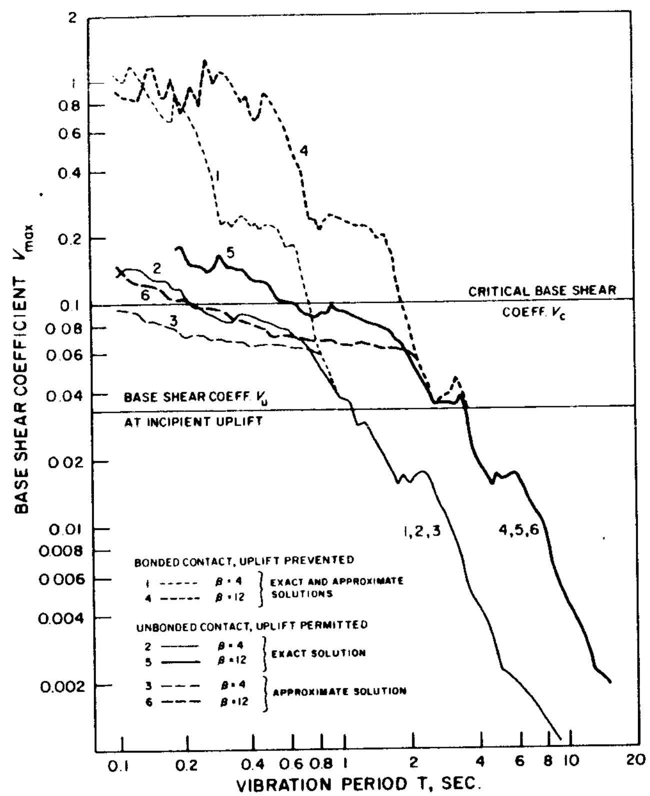

FIG. 10.—Comparison of Base Shear Response Spectra, Computed by Exact and Approximate (Eq. 25) Analysis Procedures, for Structures on Winkler Foundation Subjected to El Centro Ground Motion (h/b = **10**; $\beta \equiv \omega_v/\omega \equiv \sqrt{2bk_w/m} \div \sqrt{k/m}$ = **4 or 12**; ξ = **0.05**; $\xi_v = 2bc_w/2m\omega_v$ = **0.4**)

Eqs. 24 and 25 may provide a useful basis to determine the response of an uplifting, flexibly-supported system from the response of the corresponding uplifting, rigidly-supported system. This expectation is confirmed by the comparison presented in Figs. 9 and 10 for structures supported on two-element and Winkler foundations, respectively. The exact responses were obtained by numerically solving the governing equations for uplifting structures supported on flexible foundations by procedures presented elsewhere (6). The approximate results were obtained from Eqs. 24 and 25, in which $\tilde{S}_d$ and $(\tilde{S}_a/g)$ were obtained not from the earthquake response spectrum, but as the maximum deformation and base shear coefficient determined by analyzing the response history of uplifting, rigidly-supported systems with vibration period, $\tilde{T}$, and damping ratio, $\tilde{\xi}$, subjected to ground motion, $\ddot{u}_g(t)$, using the exact analysis procedures presented in the first part of this paper. The agreement between the two sets of results is very good for structures supported on two-element foundation and good enough to be useful in practical de-

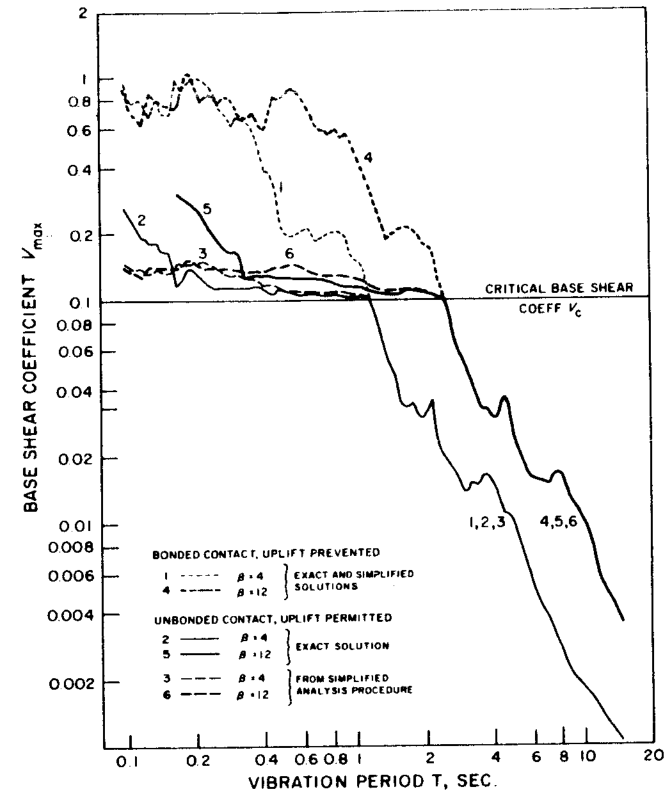

FIG. 11.—Comparison of Base Shear Response Spectra, Computed by Exact and Simplified Analysis Procedures, for Structures on Two-Element Foundation Subjected to El Centro Ground Motion (h/b = **10**; β = **4 or 12**; ξ = **0.05**; ξ_v = **0.4**)

sign applications for structures supported on Winkler foundation. That the agreement deteriorates for Winkler foundation should not be surprising because the approximate analysis can do no better than the exact analysis of the equivalent two-element system, which significantly underestimates the response of short period structures (6).

We have now seen that the response of structures supported on flexible soils with foundation mat permitted to uplift can be determined to a useful degree of accuracy by analyzing rigidly supported systems with vibration period, $\tilde{T}$, and damping ratio, $\tilde{\xi}$, and using Eqs. 24 and 25. The rigidly supported system was analyzed exactly by solving Eqs. 1 and 2, subject to the transition criteria of Eqs. 5 and 8, to obtain the results presented in Figs. 9 and 10. The more convenient alternative is to use the simplified analysis procedure which led to Eqs. 16 and 17 to analyze the rigidly supported system. The results obtained in this manner are presented in Figs. 11 and 12. Except for the very short period range, where the additional approximation (introduced by use of Eq. 16) yields results that are consistently lower than the exact values, the degree of accuracy is maintained over a wide range of fixed-base vibration periods for structures supported on two-element foundation. For structures supported on Winkler foundation, the additional approximation leads to further deterioration in the accuracy.

Structures on Viscoelastic Half-Space Foundation

Rigorous earthquake response analysis of structures supported through a foundation mat resting on more realistic models of the foundation soil, such as a viscoelastic half-space, must be very complex if the foundation mat uplifts during an earthquake. In addition to the nonlinearity introduced by uplifting, the analysis is further complicated by the frequency dependence of the stiffness and damping coefficients for a foundation mat supported on a viscoelastic half-space. Because of these complexities, exact analysis procedures have not been developed for this class of systems with the foundation mat permitted to uplift. However, the dynamics of structures with foundation mat firmly bonded to the sup-

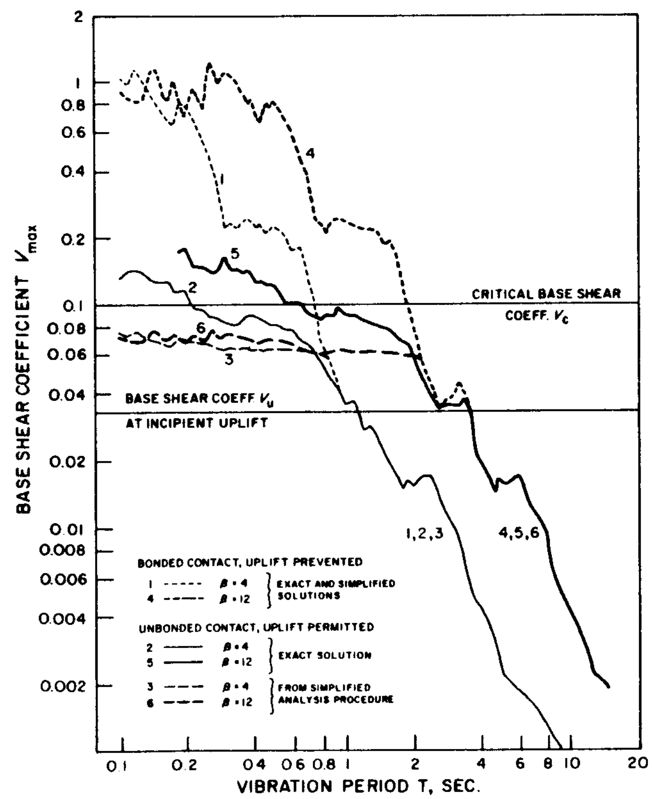

FIG. 12.—Comparison of Base Shear Response Spectra, Computed by Exact and Simplified Analysis Procedures, for Structures on Winkler Foundation Subjected to El Centro Ground Motion (h/b = 10; β = 4 or 12; ξ = 0.05; ξ_v = 0.4)

porting viscoelastic half-space has been the subject of many investigations (see Ref. 5 and its references).

In the simplified analysis procedure developed in the preceding section, the maximum response of a structure having a fixed-base natural period, T, and damping ratio, ξ, supported on flexible foundation soil is determined in terms of the response of a rigidly supported structure with vibration period, $\tilde{T}$, and damping ratio, $\tilde{\xi}$. Analytical basis for the relationship between the response of flexibly and rigidly supported structures was developed under the restriction of bonded foundation mat restrained from uplift. Although this relationship is strictly not valid for systems with foundation mat permitted to uplift, it was shown to provide results with useful degree of accuracy. The expressions presented for $\tilde{T}$ in Eq. 21, $\tilde{\xi}$ in Eq. 22 and $\tilde{\xi}_o$ in Eq. 23 were derived for the systems of Figs. 1(b–c) with frequency-independent foundation stiffness and damping.

The analytical basis for the relationship between the response of flexibly and rigidly supported systems, and the expressions for $\tilde{T}$, $\tilde{\xi}$, and $\tilde{\xi}_o$ presented in the preceding sections are special cases of the results obtained by Veletsos for structures supported on foundation mat bonded to a viscoelastic half-space. Extending the arguments of the preceding section, which justify this relationship even for uplifting structures, it would seem appropriate to apply the simplified procedure presented in the preceding section to uplifting structures supported on a viscoelastic half-space provided $\tilde{T}$ and $\tilde{\xi}_o$ are computed by the procedures developed by Veletsos (5) instead of Eqs. 21 and 23.

Summary and Conclusions

Starting with the earthquake response spectrum, simplified analysis procedures have been developed to consider the beneficial effects of foundation uplift in estimating the earthquake response of structures, which respond essentially as single-degree-of-freedom systems in their fixed-base condition. The simplified analysis procedures provide estimates for maximum base shear and deformation to a useful degree of accuracy for practical structural design.

With the information that has been presented, the analysis of the uplifting structure on rigid foundation soil may be implemented as follows:

1. Under the assumption that the foundation mat is bonded to the supporting soil and it does not uplift, compute the natural vibration period, T, and estimate the damping ratio, ξ, for the structure.
2. From the response spectra for single-degree-of-freedom systems subjected to the specified ground motion, evaluate the ordinate S_d of the deformation response spectrum and the ordinate $S_a = \omega^2 S_d$ of the pseudo-acceleration response spectrum, corresponding to T and ξ. The maximum base shear for the structure not permitted to uplift is equal to mS_a and the maximum deformation is equal to S_d. If this base shear is less than the critical base shear V_c, evaluated from Eq. 7, the foundation mat of the structure will not uplift during the earthquake, and the computed responses are satisfactory. Otherwise, proceed with the following steps.

3. Determine the damping ratio, ζ, of the system during uplift from Eq. 4*b*.
4. Compute the maximum base shear, V_{max}, for the uplifting structure from Eq. 16 or the simpler Eq. 17, and the corresponding maximum deformation $u_{max} = V_{max}/k$.

The simplified analysis procedure to estimate the response of structures supported on flexible foundation soil considering foundation mat uplift may be summarized as follows:

1. Under the assumption that the foundation mat is bonded to the supporting soil, determine the effective natural period of the system $\tilde{T}$ from Eq. 21 and the effective overall damping ratio, $\tilde{\xi}$, from Eqs. 22 and 23.
2. Determine the maximum base shear, V_{max} from Eq. 25, in which $\tilde{S}_a$ = the ordinate of the pseudo acceleration spectrum corresponding to $\tilde{T}$ and $\tilde{\xi}$, and the corresponding deformation from $u_{max} = V_{max}/k$ or from Eq. 24.
3. If the computed V_{max} is less than the critical base shear, V_c, evaluated from Eq. 7, the foundation mat will not uplift during the earthquake, and the computed responses are satisfactory. Otherwise, proceed with following steps.
4. Determine the damping ratio, ζ, of the system during uplift from Eq. 4*b*.
5. Compute the maximum base shear, V_{max}, for the uplifting structure from Eq. 16 or the simpler Eq. 17 with S_a replaced by $\tilde{S}_a$, and the corresponding maximum deformation, $u_{max} = V_{max}/k$. Note that these results would be the same as those computed from Eqs. 24 and 25 in which $\tilde{S}_d$ and $\tilde{S}_a/g$ are obtained as the maximum deformation and base shear coefficient from Eq. 16 (or Eq. 17).

This latter procedure can also be applied to the analysis of uplifting structures supported on a viscoelastic half-space provided $\tilde{T}$ and $\tilde{\xi}_o$ are computed by the procedures developed by Veletsos (5) instead of Eqs. 21 and 23.

Because of their simplicity, the analysis procedures presented in this paper should be useful in practical application. The foundation parameters are very difficult to evaluate in practice, because they depend on the details of the foundation design, including the degree-of-foundation embedment and base deformability. Therefore, the structural response analysis should be repeated for a range of foundation parameters to establish design values. The simplified analysis procedures presented in this paper are ideally suited for such repetitive analysis.

Although the maximum base shear is reduced because of foundation-mat uplift for structures over a wide range of fixed-base vibration periods, very short-period structures may be exceptions to this behavior. Their response may increase because of foundation mat uplift if the foundation soil is very stiff. The increased response in spite of foundation mat uplift may be of significance for slender, short-period buildings, isolated shear walls in low-rise buildings, other unusual buildings,

and industrial equipment. The simplified analysis procedure is not applicable to such slender, short-period systems.

Acknowledgment

This research was conducted under Grants CEE81-05790 and CEE84-02271 from the National Science Foundation for which the writers are grateful.

Appendix I.—Derivation of Eq. 18

The differential equations governing the small-amplitude motion of the systems of Figs. 1(*b–c*), with the foundation mat bonded to the supporting soil, can be derived by considering the lateral equilibrium of forces acting on the structural mass, m, and moment equilibrium of the entire system about the center of the foundation mat. These equilibrium conditions lead to

$$m(\ddot{u} + h\ddot{\theta} + \ddot{u}_g) + c\dot{u} + ku = 0 \quad (26)$$

$$mh(\ddot{u} + h\ddot{\theta} + \ddot{u}_g) + C_\theta\dot{\theta} + K_\theta\theta = 0 \quad (27)$$

in which K_θ and C_θ, the rocking stiffness and damping of the foundation mat, are defined in Eq. 19. These equations can be re-written in matrix notation

$$\begin{bmatrix} m & m \\ mh & mh \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ h\ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c & \\ & \dfrac{C_\theta}{h} \end{bmatrix} \begin{Bmatrix} \dot{u} \\ h\dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & \\ & \dfrac{K_\theta}{h} \end{bmatrix} \begin{Bmatrix} u \\ h\theta \end{Bmatrix} = -m\ddot{u}_g(t) \begin{Bmatrix} 1 \\ h \end{Bmatrix} \quad (28)$$

Multiplying Eq. 26 by h and subtracting it from Eq. 27 leads to

$$\frac{K_\theta}{h}(h\theta) - khu + \frac{C_\theta}{h}(h\dot{\theta}) - ch\dot{u} = 0 \quad (29)$$

For undamped systems, $c = C_\theta = 0$ and from Eq. 29

$$h\theta = \frac{kh^2}{K_\theta} u \quad (30)$$

which can be expressed as a transformation

$$\begin{Bmatrix} u \\ h\theta \end{Bmatrix} = u \begin{Bmatrix} 1 \\ \dfrac{kh^2}{K_\theta} \end{Bmatrix} \quad (31)$$

Substituting Eq. 31 in Eq. 28 and premultiplying by the transpose of the transformation matrix in Eq. 31 leads to

$$\left[1 + \frac{kh^2}{K_\theta}\right]^2 m\ddot{u} + \left[c + \frac{C_\theta}{h^2}\left(\frac{kh^2}{K_\theta}\right)^2\right]\dot{u} + k\left[1 + \frac{kh^2}{K_\theta}\right]u$$

$$= -m\left[1 + \frac{kh^2}{K_\theta}\right]\ddot{u}_g(t) \quad (32)$$

which is the same as Eq. 18. It is exactly valid only for undamped systems but not for damped systems because Eq. 30 is restricted to undamped systems.

Appendix II.—References

1. Crandall, S. H., and Mark, W. D., *Random Vibration in Mechanical Systems*, Academic Press, New York, N.Y., 1963.
2. Huckelbridge, A. A., and Clough, R. W., "Seismic Response of Uplifting Building Frame," *Journal of the Structural Division*, ASCE, Vol. 104, No. ST8, Aug., 1978, pp. 1211–1229.
3. Meek, J. W., "Effects of Foundation Tipping on Dynamic Response," *Journal of the Structural Division*, ASCE, Vol. 101, No. ST7, July, 1975, pp. 1297–1311.
4. Psycharis, I. N., "Dynamic Behavior of Rocking Structures Allowed to Uplift," *Report No. EERL 81-02*, Earthquake Engineering Research Laboratory, California Institute of Technology, Pasadena, Calif., Aug., 1981; also see *Earthquake Engineering and Structural Dynamics*, Vol. 11, 1983, pp. 57–76 and pp. 501–521.
5. Veletsos, A. S., "Dynamics of Structure-Foundation Systems," in *Structural and Geotechnical Mechanics*, W. J. Hall, ed., a volume honoring Nathan M. Newmark, Prentice-Hall, Englewood Cliffs, N.J., 1977, pp. 333–361.
6. Yim, C. S., and Chopra, A. K., "Effects of Transient Foundation Uplift on Earthquake Response of Structures," *Report No. UCB/EERC-83/09*, Earthquake Engineering Research Center, University of California, Berkeley, Calif., June, 1983, 129 pp.

TECHNICAL NOTES

Note.—Discussion open until September 1, 1985. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. This paper is part of the ***Journal of Structural Engineering***, Vol. 111, No. 4, April, 1985.