A Proofs

Proposition 1. *Given a linear 3D tensor field $T(x,y,z) = T_0 + xT_x + yT_y + zT_z$, any tensor value can occur at most once in the domain. Moreover, if a tensor value appears in the domain, its multiples cannot occur anymore in the field.*

Proof. Given a tensor t, the invertibility of $\mathfrak{F}$ implies that there cannot be multiple $(x,y,z,1,0)$ that map to the same tensor. Therefore, a tensor cannot occur multiple times in the field. Additionally, since $\mathfrak{F}^{-1}$ is linear, when we scale t by some $r \neq 1$ we get $\mathfrak{F}^{-1}(rt) = r\mathfrak{F}^{-1}(t) = (rx, ry, rz, r, 0)$, so its w coordinate must be $r \neq 1$. Since w must be 1 for a tensor to occur in the field, rt cannot occur in the field. □

Theorem 2. *Let t be a degenerate tensor that occurs in a linear tensor field $T(x,y,z) = T_0 + xT_x + yT_y + zT_z$, and v be its dominant eigenvector. Then $v^T \overline{T} v = 0$.*

Proof. Since t is degenerate, it has the form $t = k(vv^T - \frac{\mathbb{I}}{3})$ for some $k \in \mathbb{R}$. For t to occur in the field, we need $u = \mathfrak{T}^{-1}(t) = 0$. Therefore,

$$u = \langle \overline{T}, t \rangle = 0 \tag{22}$$

$$\text{trace}[k(vv^T - \frac{\mathbb{I}}{3})\overline{T}] = 0 \tag{23}$$

This is equivalent to

$$\text{trace}(vv^T\overline{T} - \frac{\overline{T}}{3}) = 0 \tag{24}$$

or

$$\text{trace}(vv^T\overline{T}) = \frac{1}{3}\,\text{trace}(\overline{T}) \tag{25}$$

Since $\overline{T}$ is traceless, the above equation becomes

$$\text{trace}(vv^T\overline{T}) = 0 \tag{26}$$

Applying the *cyclic property* of trace [7], we have

$$\text{trace}(v^T\overline{T}v) = \text{trace}(vv^T\overline{T}) = 0 \tag{27}$$

Since $v^T\overline{T}v$ is a 1×1 matrix, we have

$$v^T\overline{T}v = \text{trace}(v^T\overline{T}v) = 0 \tag{28}$$

□

Theorem 3. *Given a linear tensor field $T(x,y,z) = T_0 + xT_x + yT_y + zT_z$ and a unit vector v that satisfies $v^T\overline{T}v = 0$, there exist $x_0, y_0, z_0 \in \mathbb{R}$ such that $T(x_0, y_0, z_0)$ is a degenerate tensor and v is a dominant eigenvector of $T(x_0, y_0, z_0)$. The dominant eigenvalue is given by $k = \frac{1}{v^T T_0' v}$.*

Proof. Note that v is the dominant eigenvector of the tensor $t = k(vv^T - \frac{\mathbb{I}}{3})$, for all $k \neq 0$. We must choose k so that t occurs in the field. Since $v^T\overline{T}v = 0$, we have $0 = kv^T\overline{T}v = \text{trace}(kv^T\overline{T}v) = \text{trace}(kvv^T\overline{T}) = \text{trace}(kvv^T\overline{T}) - \text{trace}(k\frac{\mathbb{I}}{3}\overline{T}) = \text{trace}(k(vv^T - \frac{\mathbb{I}}{3})\overline{T}) = \langle \overline{T}, t \rangle$. This implies that $\mathfrak{T}^{-1}(t)$ gives $u = 0$. Next, we must ensure that $w = 1$.

$$\begin{aligned} 1 = w &= \langle T_0', t \rangle && (29)\\ &= \text{trace}[k(vv^T - \frac{\mathbb{I}}{3})T_0'] && (30)\\ &= k\,\text{trace}(vv^T T_0') - \frac{k}{3}\,\text{trace}(T_0') && (31)\\ &= k\,\text{trace}(vv^T T_0') && (32)\\ &= kv^T T_0' v && (33) \end{aligned}$$

Therefore,

$$k = \frac{1}{v^T T_0' v} \tag{34}$$

With k set to this value, t occurs in the field. This implies that k is unique. □

Theorem 4. *$\forall w, x, y, z \in \mathbb{R}$, let v_1, v_2, and v_3 be respectively the major, medium, and minor eigenvectors of a neutral tensor $t = wT_0 + xT_x + yT_y + zT_z$. Then $v_1^T\overline{T}v_1 = v_3^T\overline{T}v_3$.*

Proof. Recall that $\overline{T}$ has a zero dot product with $t = wT_0 + xT_x + yT_y + zT_z$, i.e., $\text{trace}(t\overline{T}) = 0$. Since t is neutral, it has the form $t = k(v_1v_1^T - v_3v_3^T)$ for some k. Consequently,

$$\text{trace}[(v_1v_1^T - v_3v_3^T)\overline{T}] = 0 \tag{35}$$

This is equivalent to

$$\text{trace}(v_1v_1^T\overline{T}) = \text{trace}(v_3v_3^T\overline{T}) \tag{36}$$

Reusing the *cyclic property* of trace, we obtain

$$\text{trace}(v_1^T\overline{T}v_1) = \text{trace}(v_3^T\overline{T}v_3) \tag{37}$$

Again, both $v_1^T\overline{T}v_1$ and $v_3^T\overline{T}v_3$ are 1×1 matrices, we have $v_1^T\overline{T}v_1 = v_3^T\overline{T}v_3$. □

Theorem 5. *Given a medium eigenvector v_2 which resides on the k-th level set of $v^T\overline{T}v$, the corresponding major and minor eigenvectors must reside on the $-\frac{k}{2}$-th level set of $v^T\overline{T}v$.*

Proof. Notice that $v_1^T\overline{T}v_1 + v_2^T\overline{T}v_2 + v_3^T\overline{T}v_3 = \text{trace}(V^T\overline{T}V)$ where $V = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. We have $\text{trace}(V^T\overline{T}V) = \text{trace}(\overline{T}) = 0$ since $\overline{T}$ is traceless. Consequently,

$$v_1^T\overline{T}v_1 + v_2^T\overline{T}v_2 + v_3^T\overline{T}v_3 = 0 \tag{38}$$

Since $v_1^T\overline{T}v_1 = v_3^T\overline{T}v_3$ (Theorem 4), we have that $2v_1^T\overline{T}v_1 + v_2^T\overline{T}v_2 = 0$, i.e., $v_1^T\overline{T}v_1 = v_3^T\overline{T}v_3 = -\frac{v_2^T\overline{T}v_2}{2}$. □

Theorem 6. *Under the structurally stable condition that $\overline{T}$ is non-degenerate, a level set of $v^T\overline{T}v$ on the unit sphere must be two non-intersecting non-circular spherical ellipses, except one situation where it is the union of two great circles, residing in two intersecting planes.*

Proof. Since the unit sphere remains the same under any orthonormal change of basis, we can find such a basis under which $\overline{T}$ is diagonal, i.e., $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}$ where $a \geq b \geq 0$ and $a > 0$.

Under this basis, it is straightforward to verify that a level set of $v^T\overline{T}v$ on the unit sphere is the intersection of

$$a\alpha^2 + b\beta^2 - (a+b)\gamma^2 = k \tag{39}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{40}$$

which is equivalent to

$$(2a+b)\alpha^2 + (a+2b)\beta^2 = k + (a+b) \tag{41}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{42}$$

which is the union of two ellipses (thus planar) with inversive symmetry. Additionally, we get the equations

$$(a-b)\alpha^2 - (a+2b)\gamma^2 = k - b \tag{43}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{44}$$

When $k = b$ the level set satisfies either

$$\alpha = \sqrt{\frac{a+2b}{a-b}}\gamma \tag{45}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{46}$$

or

$$\alpha = -\sqrt{\frac{a+2b}{a-b}}\gamma \tag{47}$$
$$\alpha^2+\beta^2+\gamma^2 = 1 \tag{48}$$

These are the intersections of the unit sphere with two intersecting planes, both of which contain the origin. Consequently, the level set consists of two great circles. □

Proposition 7. *Given a unit vector v, the tensor at (x,y,z) has v as its medium eigenvector if and only if*

$$M\begin{bmatrix} x\\ y\\ z\end{bmatrix} = -T_0 v \tag{49}$$

where M is the matrix $\begin{bmatrix} T_x v & T_y v & T_z v\end{bmatrix}$.

Proof. A traceless tensor $T(x,y,z)$ is neutral and has medium eigenvector v if and only if

$$T(x,y,z)v = 0. \tag{50}$$

Substituting the field, we get

$$(T_0 + xT_x + yT_y + zT_z)v = 0 \tag{51}$$
$$xT_x v + yT_y v + zT_z v = -T_0 v. \tag{52}$$

which can be rewritten as Equation 49. □

Theorem 8. *Given a unit vector v where the projection of $\overline{T}$ onto the plane orthogonal to v is a degenerate two-dimensional tensor, the set of points on the neutral surface that have v as their medium eigenvector is a line.*

Proof. Usually Equation 49 gives a unique point (x,y,z) for each v. However, if M is singular then this fails. If M is singular and $T_0 v$ is not in its image then this gives an infinity point of the neutral surface. $T_0 v$ is in its image so there is a line of possible v. □

B Singularity Line Formula

In this section we provide the detail of finding the 3D coordinates of a neutral point corresponding to one of the singularities in the medium eigenvector manifold (Section 5.2).

Let O be a singularity of a linear tensor field $T_0 + xT_x + yT_y + zT_z$ whose corresponding medium eigenvector is s. Since O corresponds to a topological circle (a straight line in $\mathbb{R}^3$), we need an additional parameter u, a unit vector perpendicular to s, to identify individual points on the line.

From Proposition 7, we know that the 3D coordinates (x,y,z) of a neutral point can be found from a given medium eigenvector v by

$$\begin{bmatrix} x\\ y\\ z\end{bmatrix} = -\begin{bmatrix} T_x v & T_y v & T_z v\end{bmatrix}^{-1} T_0 v \tag{53}$$

Thus, the point on the infinite line that corresponds to the vector u is the following limit:

$$-\lim_{\eta\to 0}\begin{bmatrix} T_x(s+\eta u) & T_y(s+\eta u) & T_z(s+\eta u)\end{bmatrix}^{-1} T_0(s+\eta u) \tag{54}$$

For convenience, we name the following variables:

$$M_s = \begin{bmatrix} T_x s & T_y s & T_z s\end{bmatrix} \tag{55}$$
$$M_u = \begin{bmatrix} T_x u & T_y u & T_z u\end{bmatrix} \tag{56}$$
$$v_s = T_0 s \tag{57}$$
$$v_u = T_0 u \tag{58}$$

Consequently, the limit in Equation 54 can be rewritten as

$$-\lim_{\eta\to 0}(M_s+\eta M_u)^{-1}(v_s+\eta v_u) \tag{59}$$

This is equivalent to

$$-\lim_{\eta\to 0}\frac{adj(M_s+\eta M_u)(v_s+\eta v_u)}{|M_s+\eta M_u|} \tag{60}$$

where $adj(M)$ and $|M|$ are the adjoint matrix and the determinant of the matrix M, respectively.

It can be verified that

$$\lim_{\eta\to 0} adj(M_s+\eta M_u)(v_s+\eta v_u) = adj(M_s)v_s = 0 \tag{61}$$

and that

$$\lim_{\eta\to 0}|M_s+\eta M_u| = |M_s| = 0 \tag{62}$$

Consequently, to evaluate the limit in Equation 60 we apply L'hospital's rule, i.e.

$$-\frac{\lim_{\eta\to 0}\frac{d}{d\eta}[adj(M_s+\eta M_u)(v_s+\eta v_u)]}{\lim_{\eta\to 0}\frac{d}{d\eta}|M_s+\eta M_u|} \tag{63}$$

From classical matrix calculus results, we have

$$\lim_{\eta\to 0}\frac{d}{d\eta}|M_s+\eta M_u| = trace(adj(M_s)M_u) \tag{64}$$

and

$$\lim_{\eta\to 0}\frac{d}{d\eta}[adj(M_s+\eta M_u)(v_s+\eta v_u)] = \Gamma v_s + adj(M_s)v_u \tag{65}$$

where

$$\Gamma = \lim_{\eta\to 0}\frac{d}{d\eta}[adj(M_s+\eta M_u)] = M_s^{-1}[trace(adj(M_s)M_u)\mathbb{I} - M_u adj(M_s)]v_s + adj(M_s)v_u \tag{66}$$

C Higher-Resolution Images

In Figure 12 we show higher-resolution images of Figure 11 (a-d).

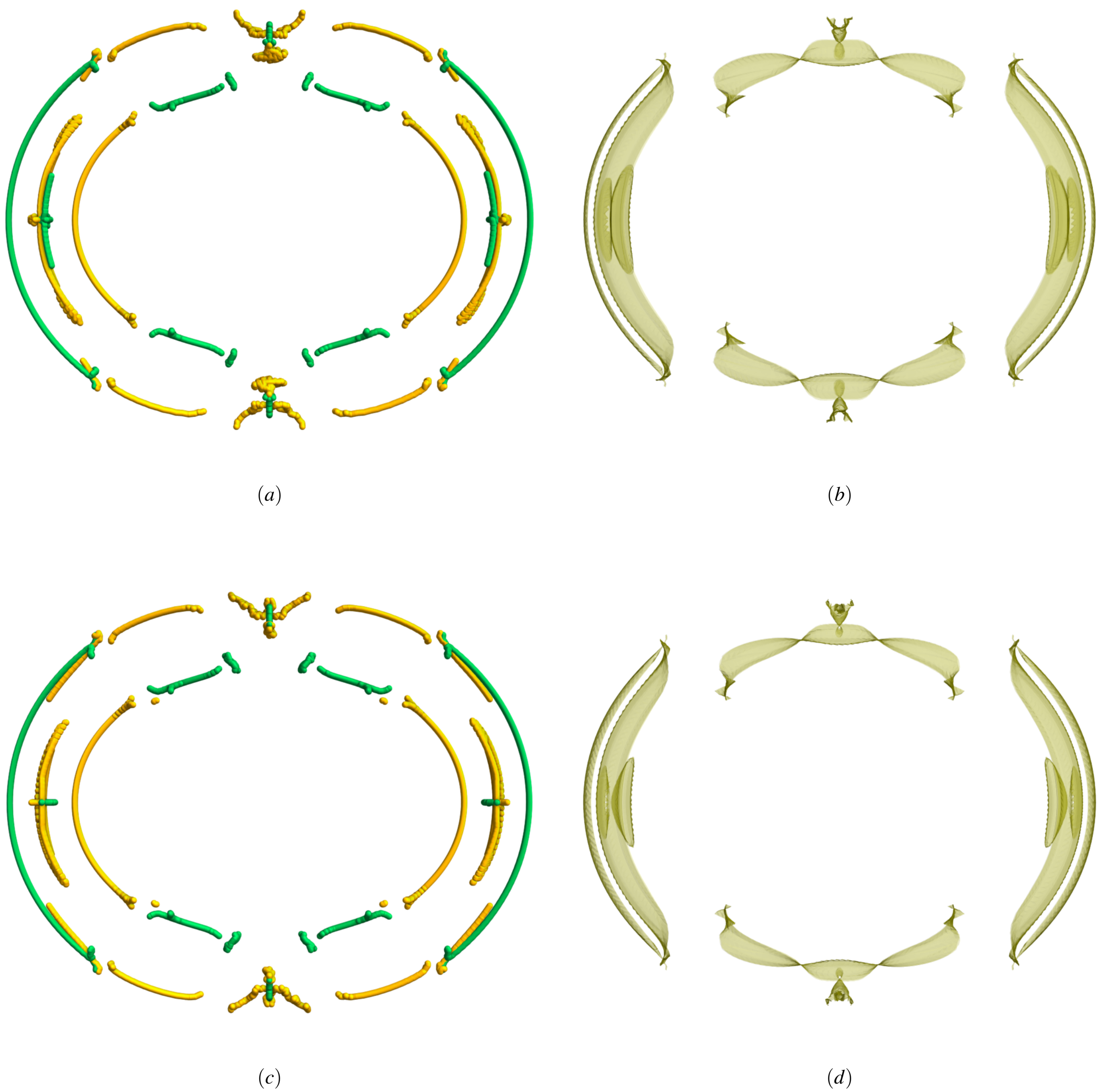

Fig. 12: Higher-resolution images of Figure 11 (a-d).