CS 514 Algorithms

Spring 2025

Dynamic Programming 101

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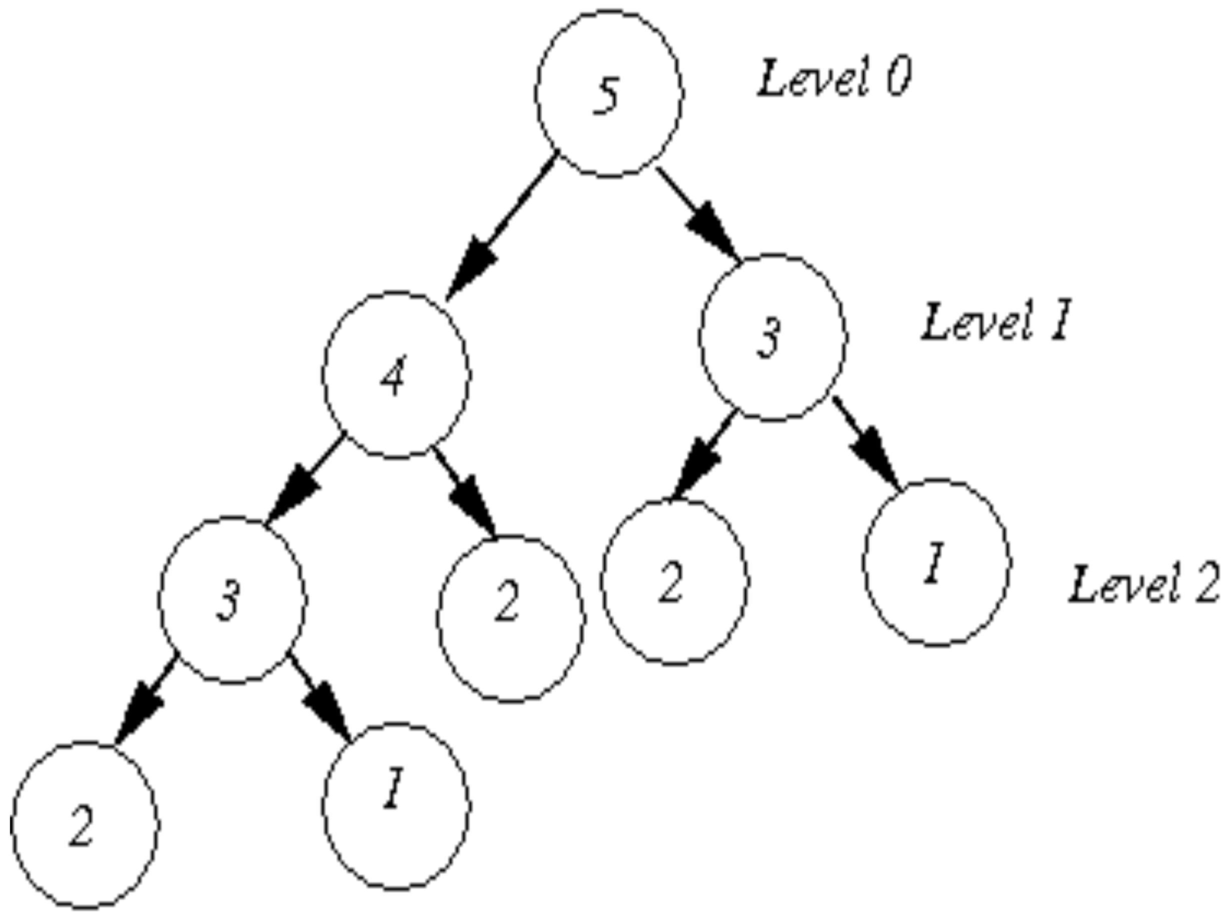

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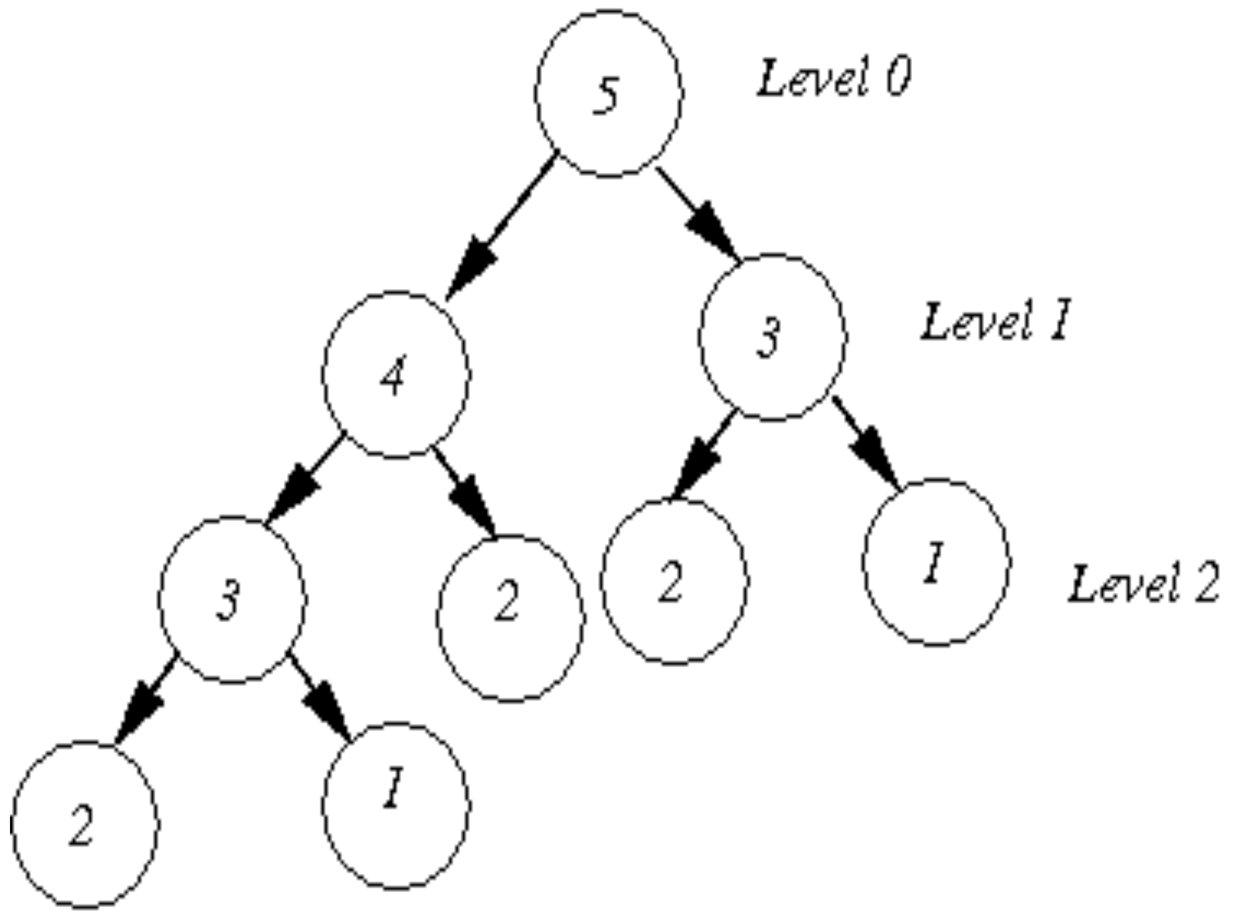

top-down with recursion: $O(1.618...^n)$

```
def fib(n):
    if n <= 2:
        return 1
    return fib(n-1) + fib(n-2)
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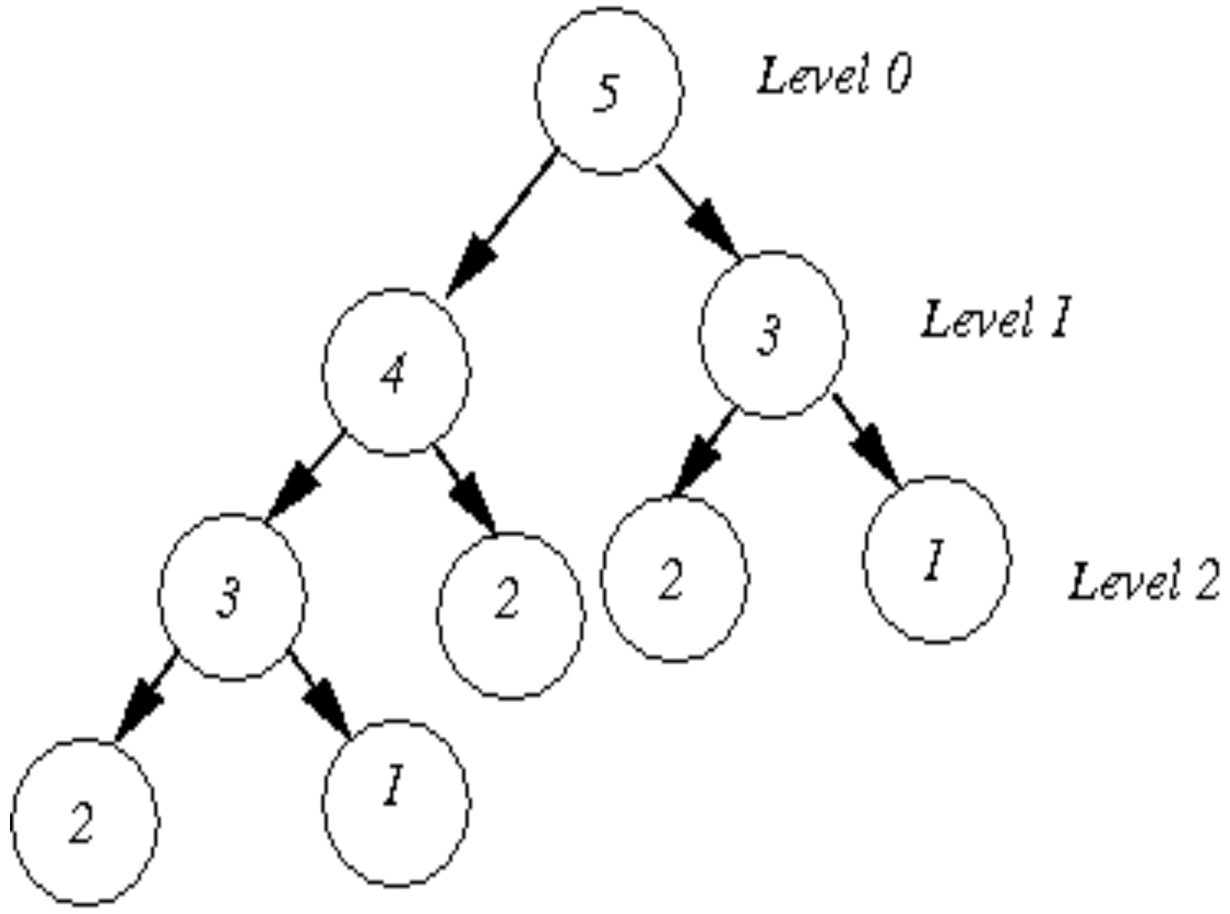

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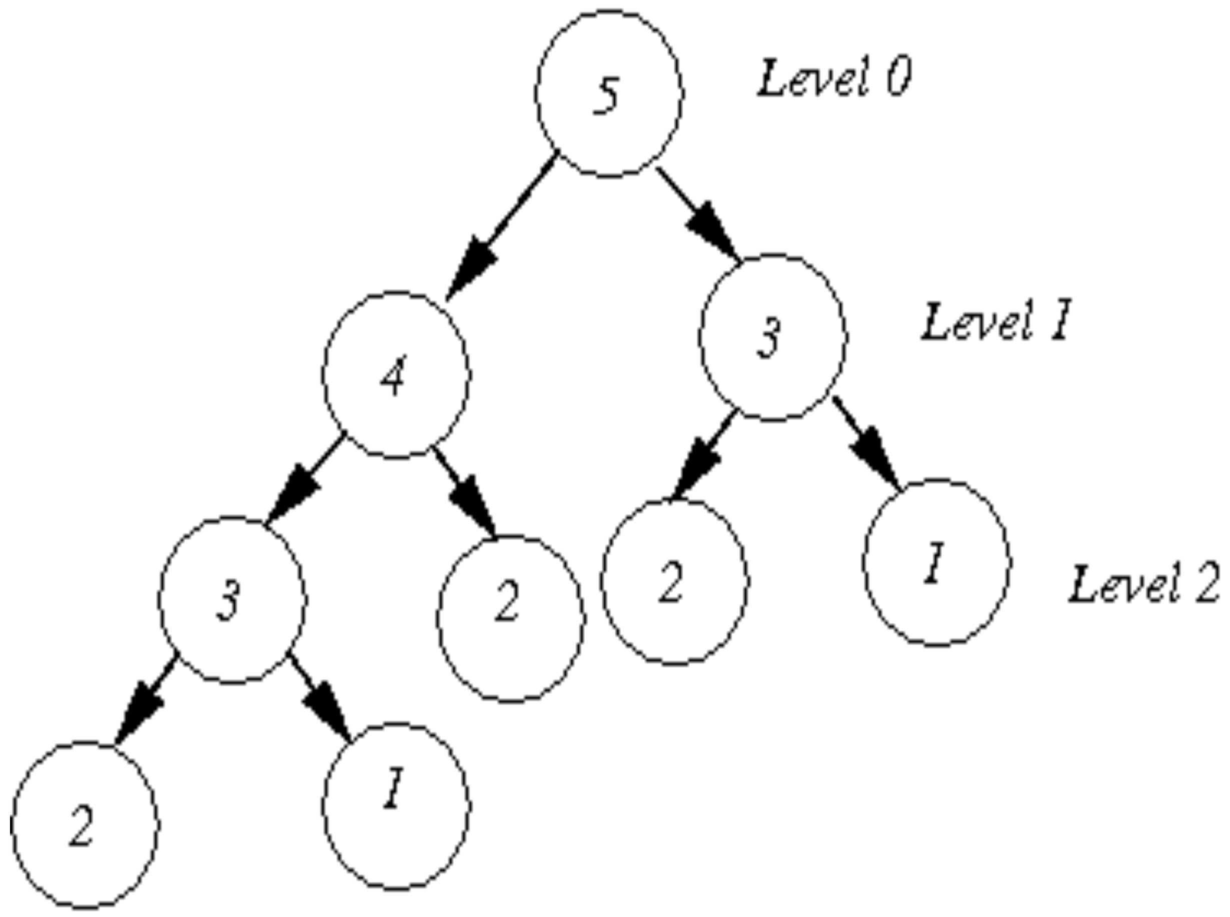

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bottom-up with memoization: $O(n)$

```
def fib0(n):
  f = [1, 1]
  for i in range(3, n+1):
    fibs.append(f[-1]+f[-2])
  return f[-1]
```

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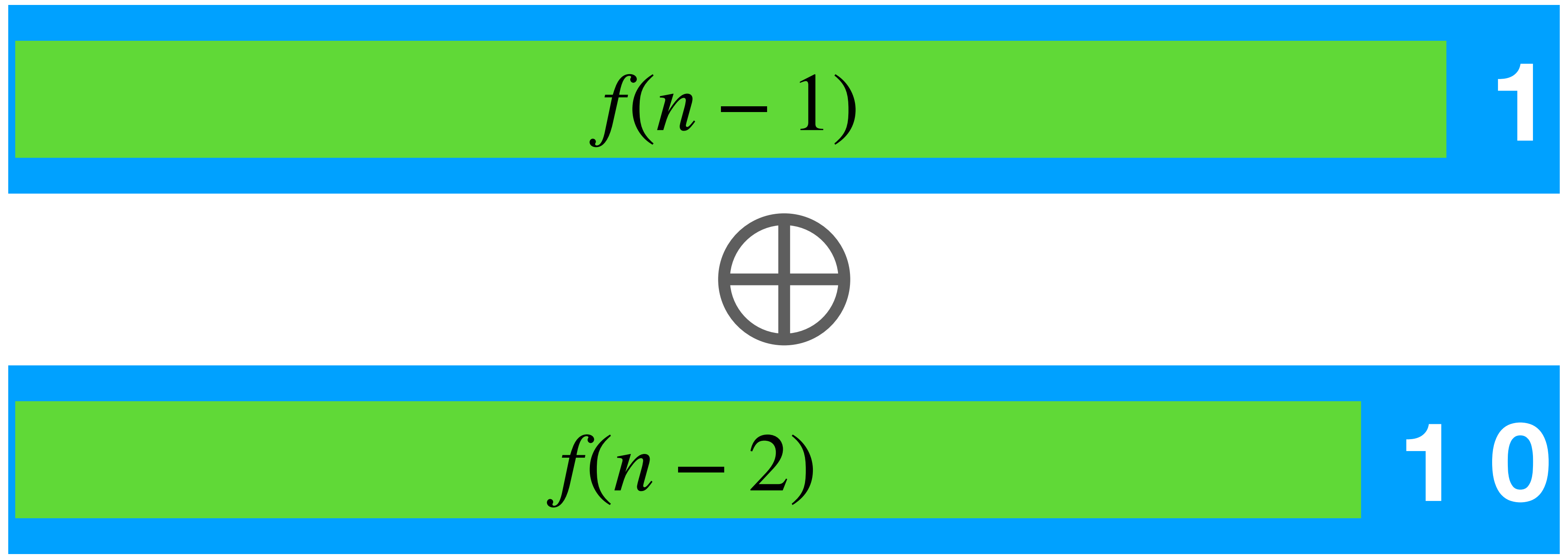

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  - last bit “1” followed by *f*(*n*-1) substrings
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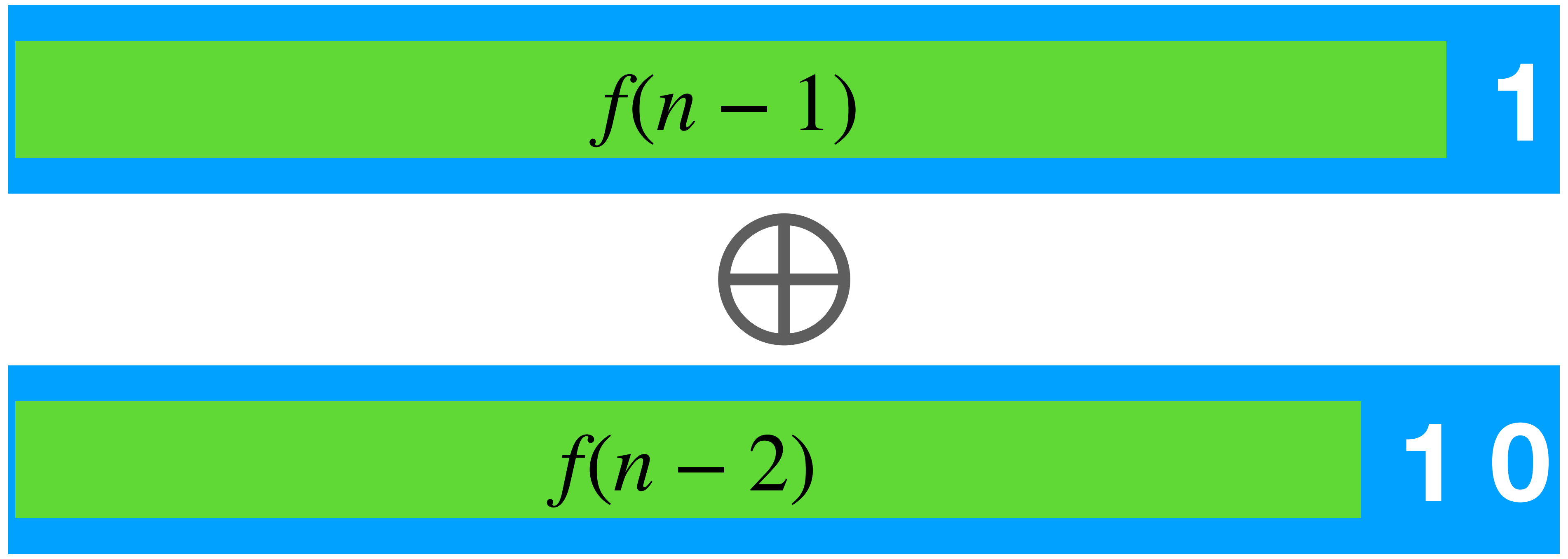

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- brute-force / exhaustive search
  - enumerate all possible (sub)sets, and evaluate independent ones
  - complexity: $O(2^n)$

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  - top-down recursion without memoization
  - complexity: $O(1.618\ldots^n)$

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f(*i*)								

best value

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- bottom-up with memoization
- complexity: $O(n)$

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S(i)	{}	{}	{9}	{10}	{9,8}	{9,8}	{9,8,2}	{9,8,4}	*best set*

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best value

backpointer

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recursively backtrack
the optimal solution

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MIS

$$f(n) = \max \begin{cases} f(n-1) + 0 \\ f(n-2) + a[n] \end{cases}$$

bitstrings

$$f(n) = + \begin{cases} f(n-1) \times 1 \\ f(n-2) \times 1 \end{cases}$$

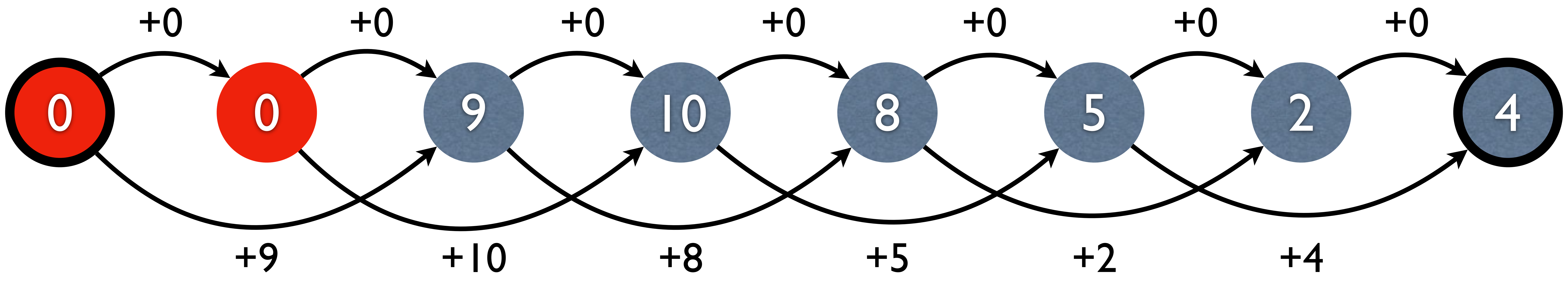
cost

reward

summary operator ⊕ (across divides)

combination operator ⊗ (within a divide)

Graph Interpretation of DP



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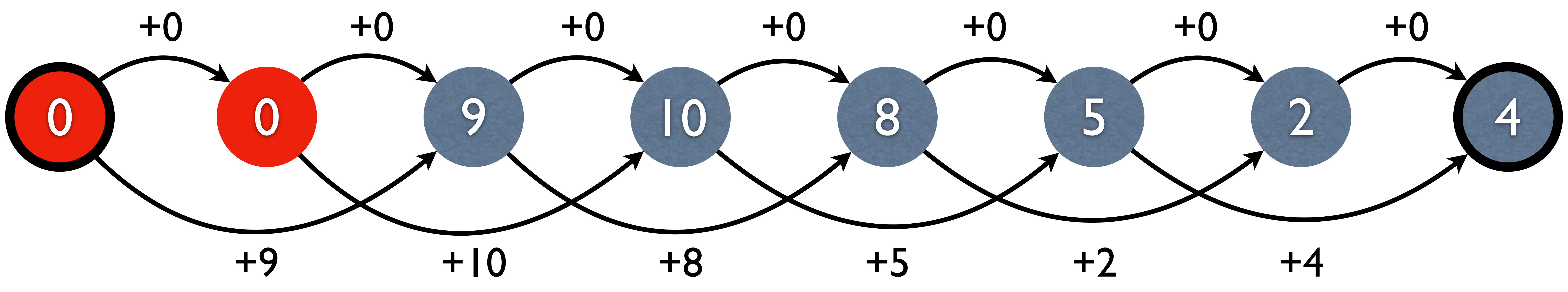

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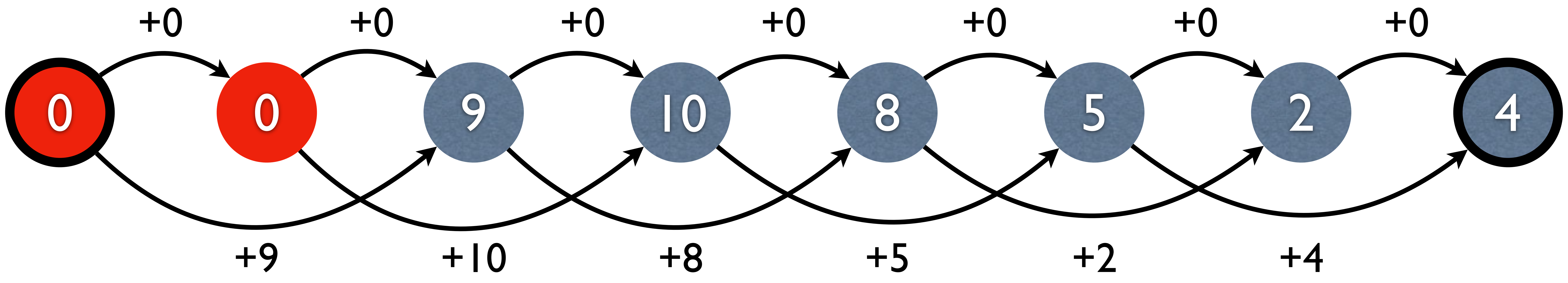

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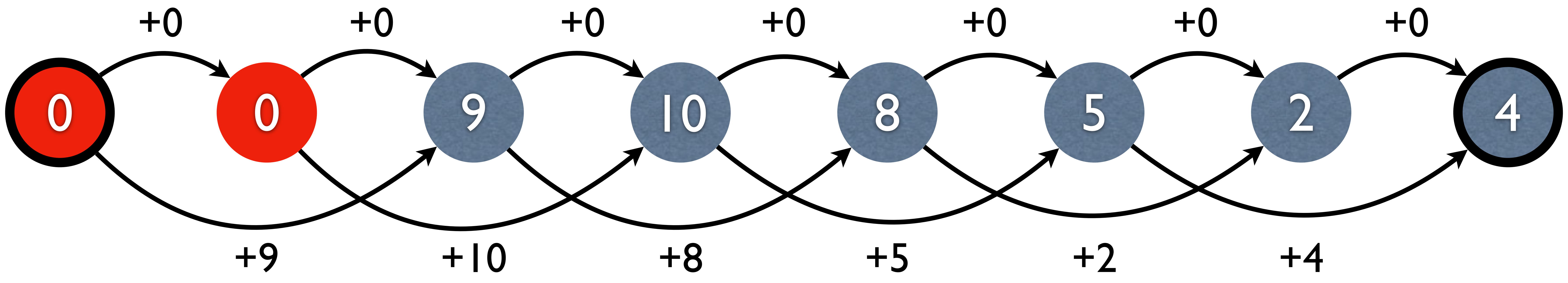

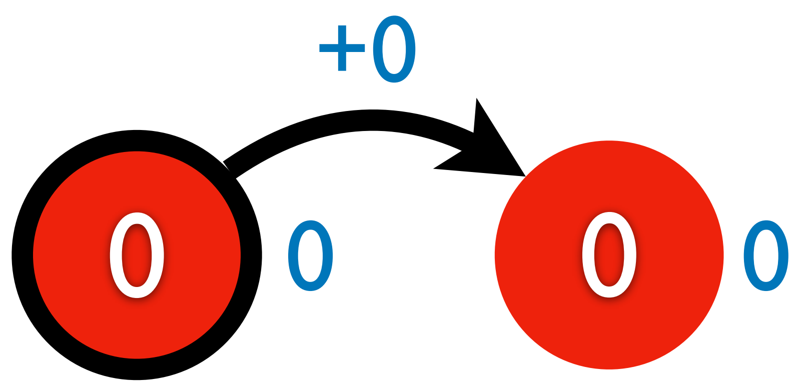

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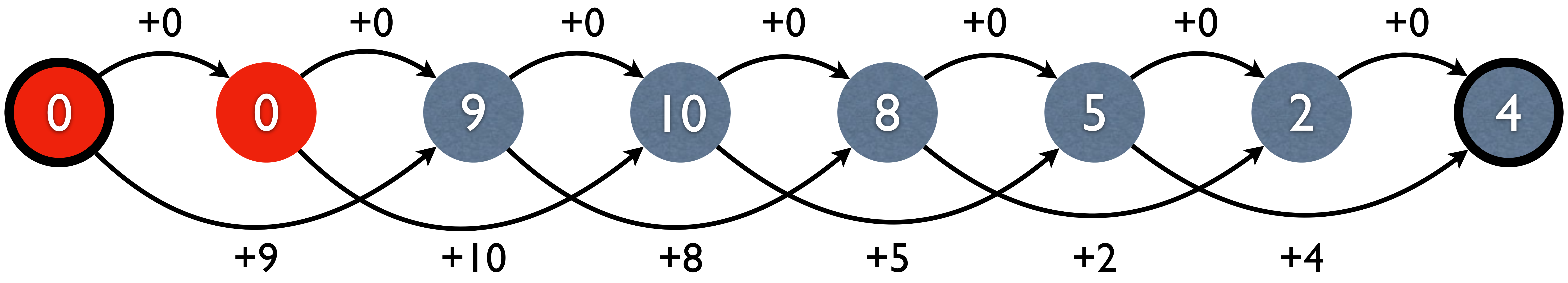

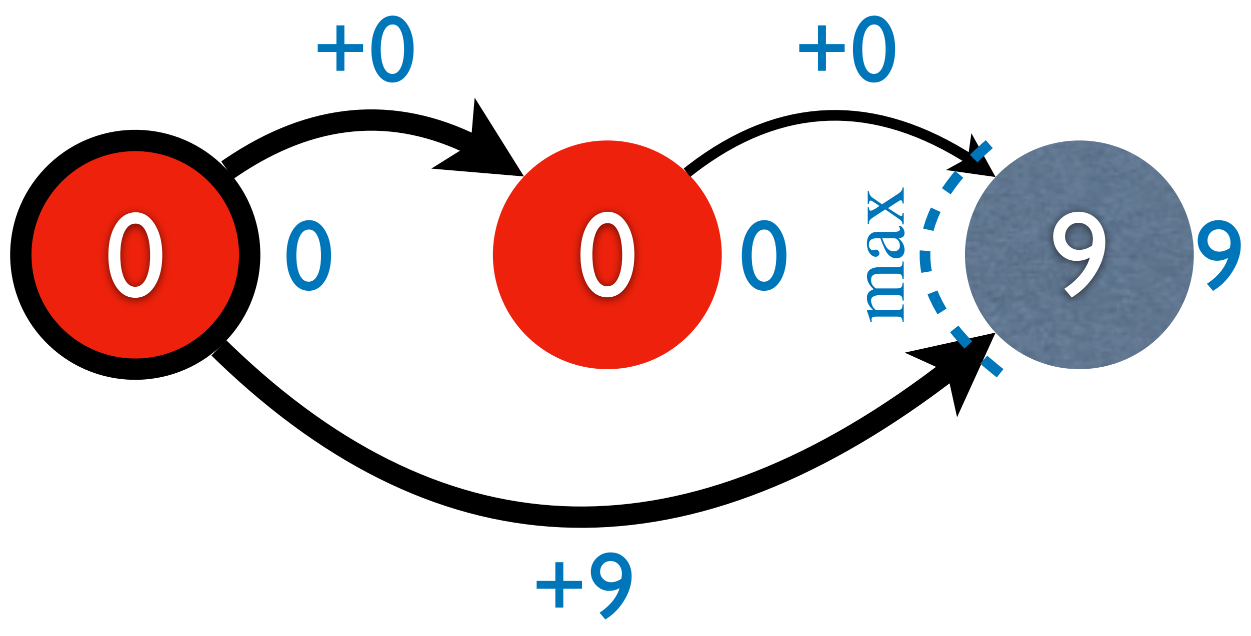

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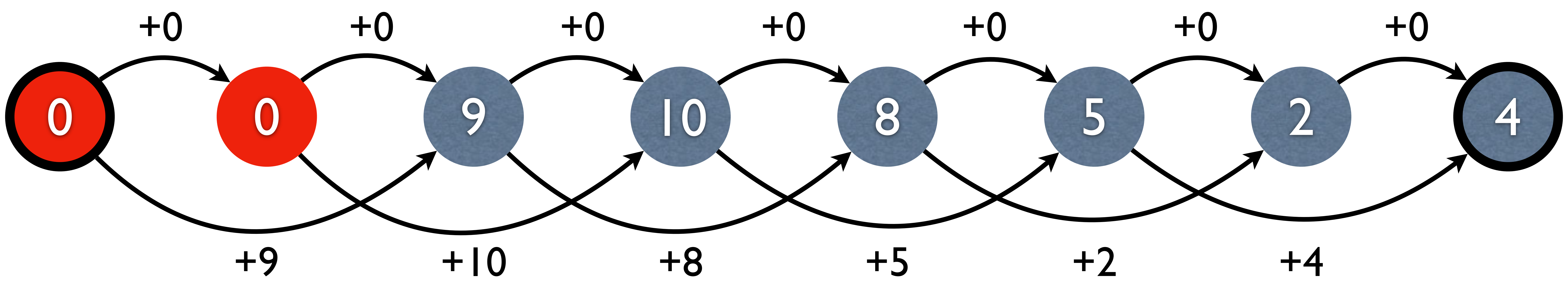

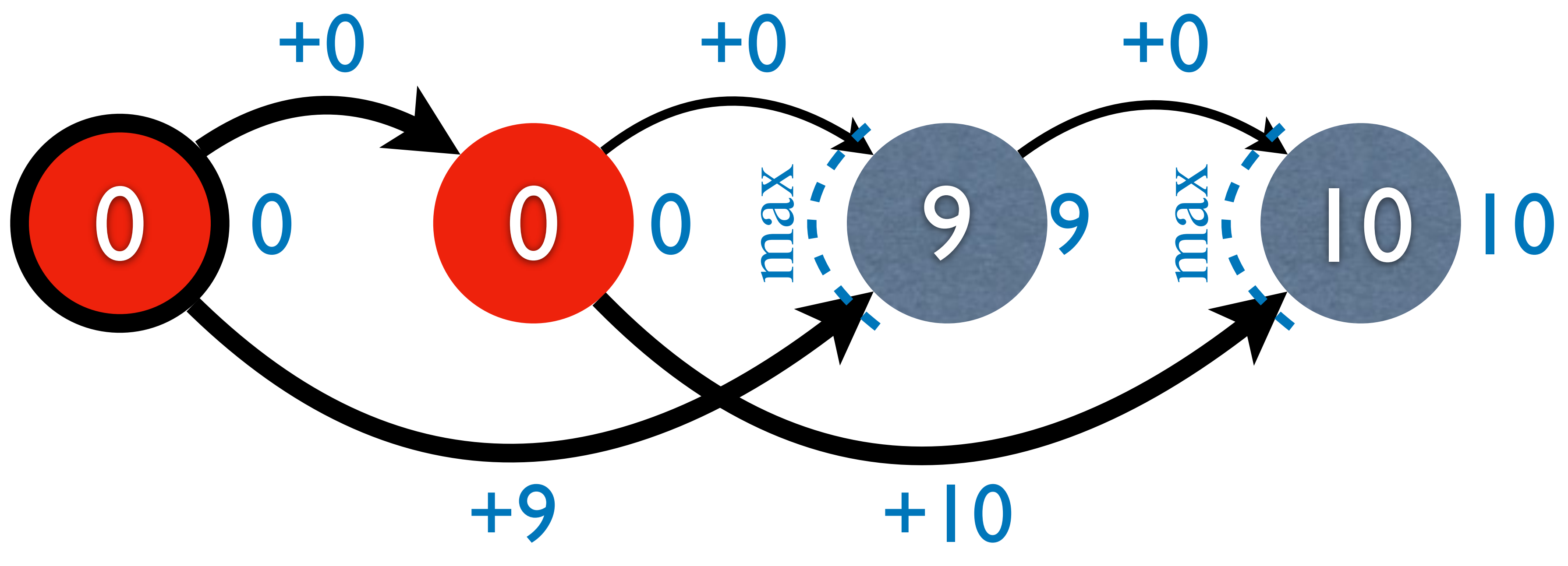

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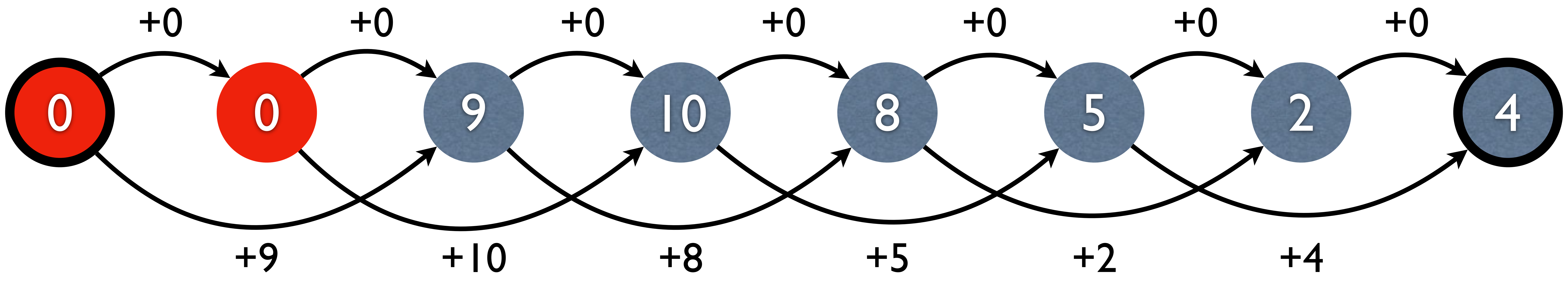

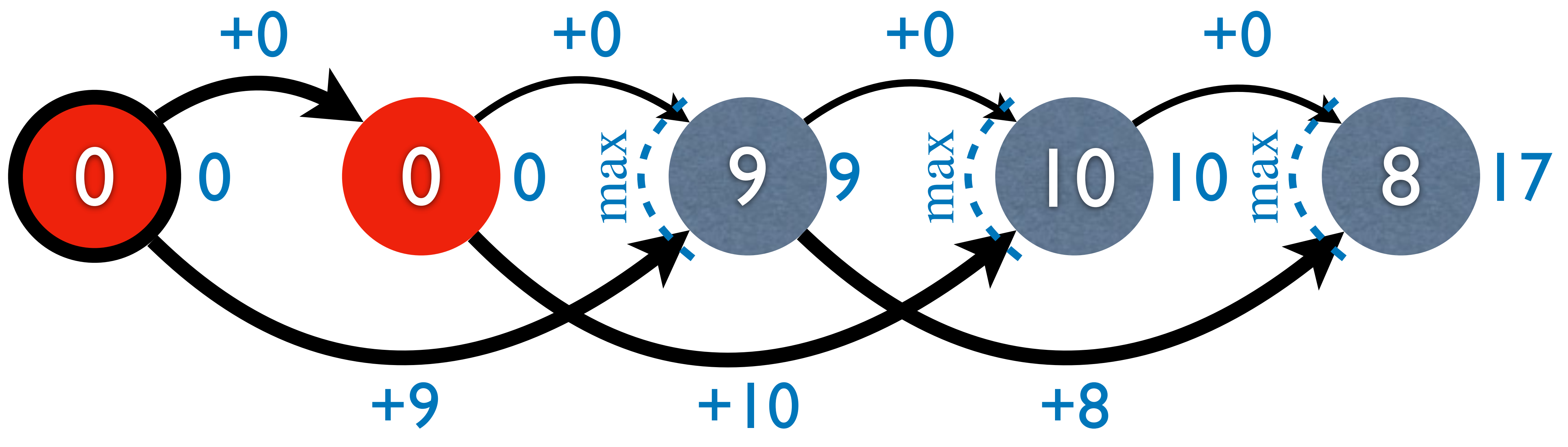

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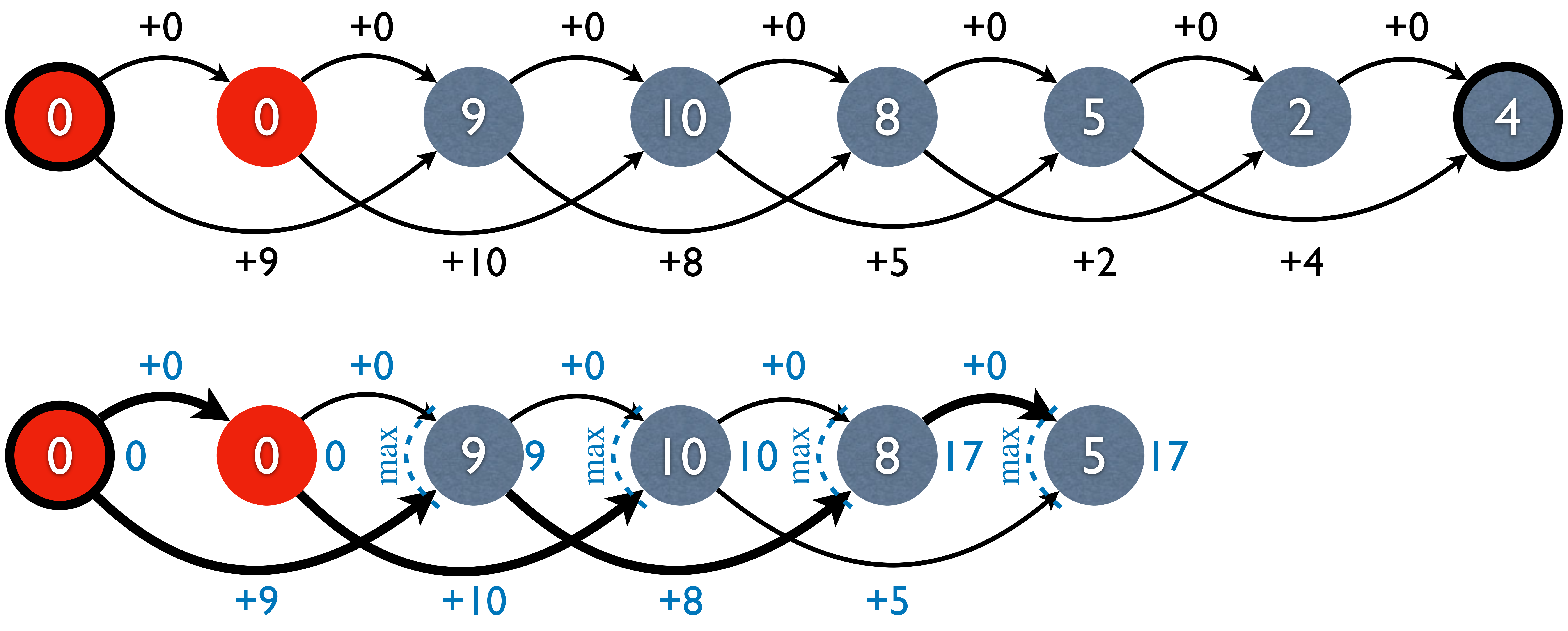

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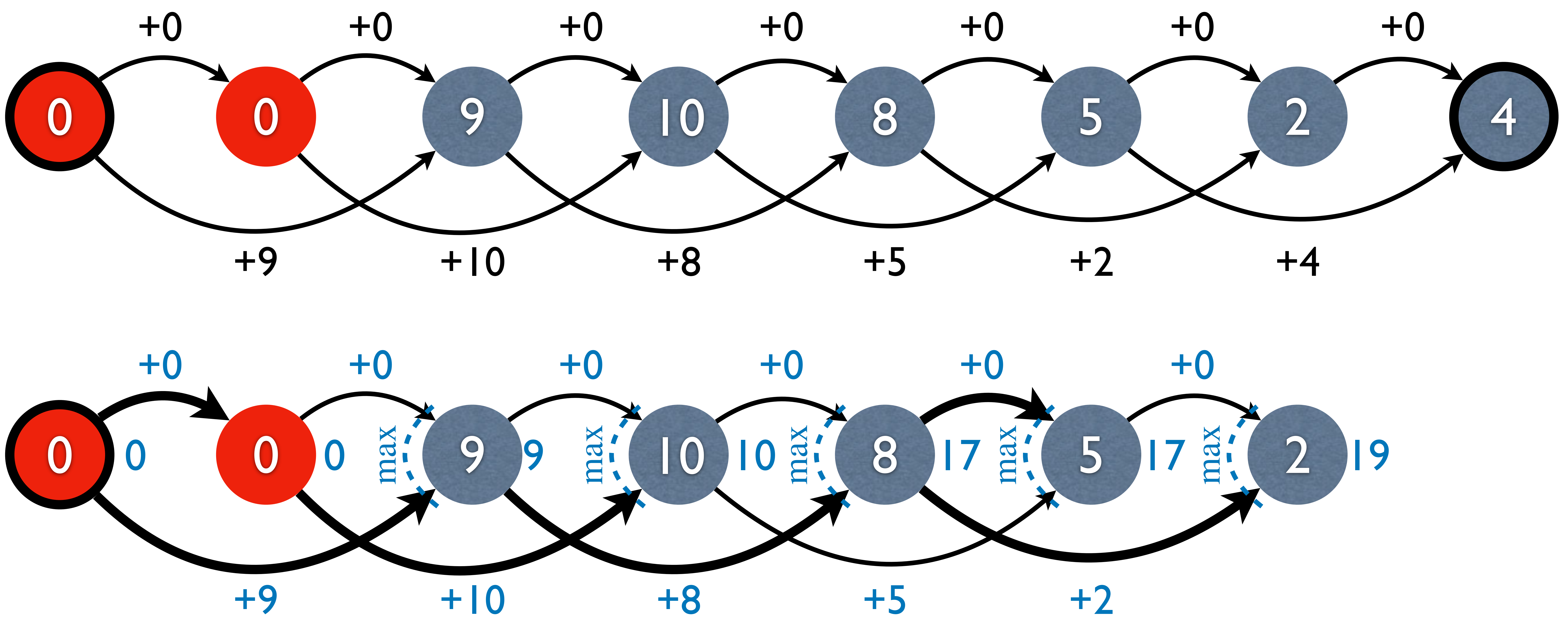

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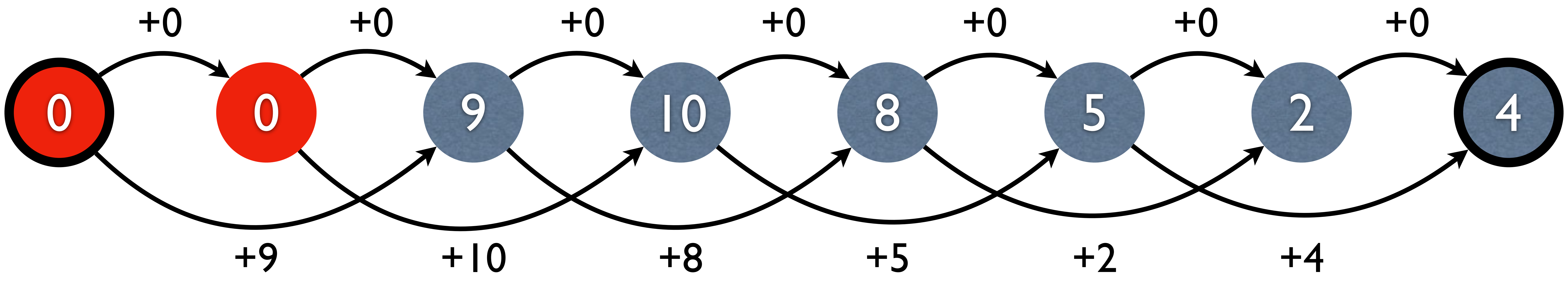

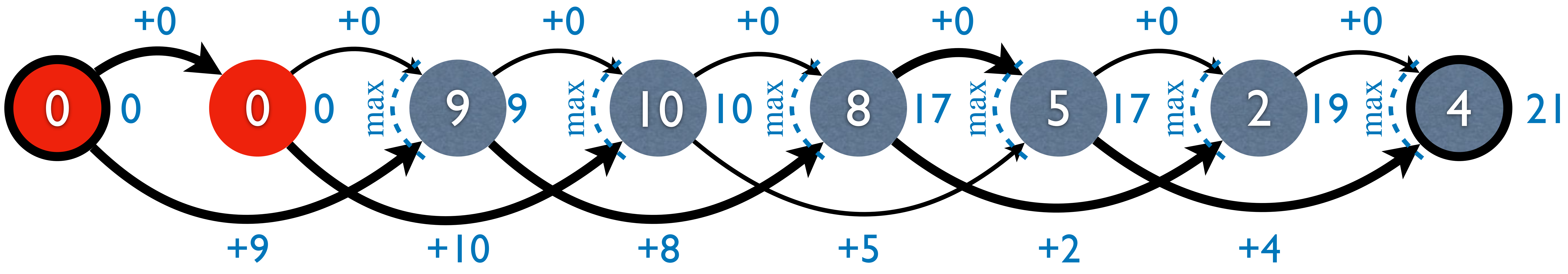

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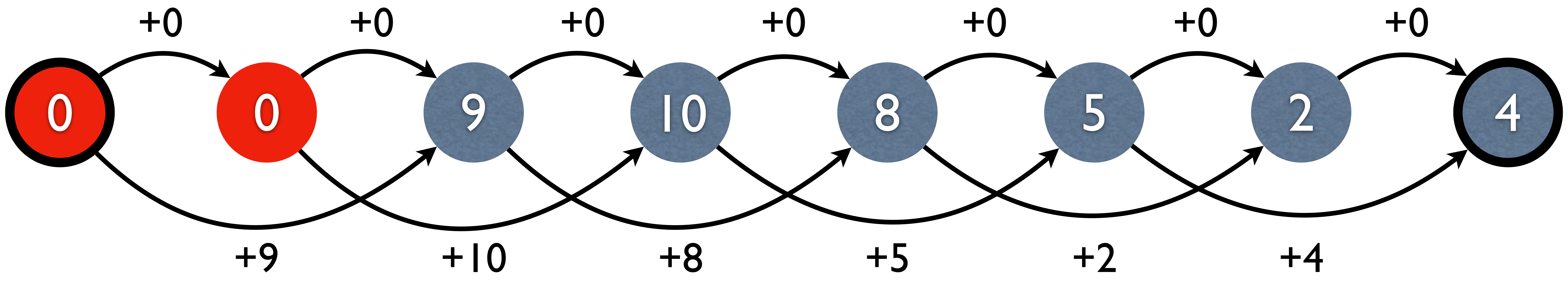

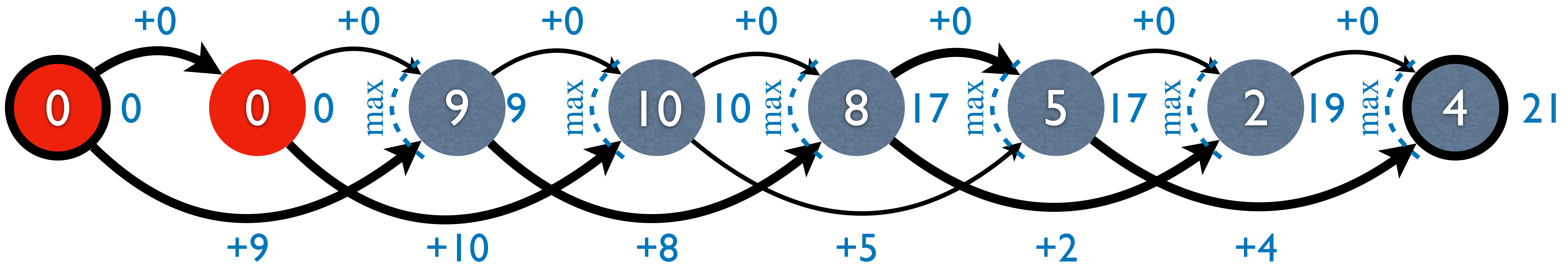

Summary

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- Dynamic Programming = multiple divides + memoized conquer + summarized combine
- overlapping subproblems due to multiple divides (still disjoint subproblems within each divide)
- two implementation styles
  - 1. recursive top-down + memoization
  - 2. bottom-up
- backtracking to recover best solution for optimization problems
  - 1. backpointers (recommended); 2. store subsolutions (not recommended — often slows down); 3. recompute on-the-fly
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summary operator ⊕ (across divides)

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